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Hyperbolic surfaces: orthospectrum, simple orthospectrum and limit set

Surfaces hyperboliques : orthospectre, orthospectre simple et ensemble limite

sous la direction de Federica Fanoni et Stéphane Sabourau

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Jaune comme la vie d'un aéroport

*«Jeune homme lève-toi, bats-toi
L'avenir appartient à celui qui s'impose
Et qui ne baisse pas les bras
Même si t'en a assez plus tu dors sur toi même
Et plus il est difficile de se relever »
- Neg Marrons*

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Abstract

This thesis is dedicated to the study of hyperbolic geometry on orientable surfaces. Two different points of view are adopted along it. First, we investigate finite-type surfaces with boundary through the orthospectrum and the simple orthospectrum. Then, we study infinite-type surfaces and in part flute surfaces through their limit set.

In the first part, we characterize the rigidity of the orthospectrum and simple orthospectrum: we show that only a finite number of surfaces up to isometry have the same orthospectrum. For surfaces with one boundary component, we show that this statement is also true for the simple orthospectrum. Moreover, we show that generic surfaces are determined by their orthospectrum and generic surfaces with one boundary component are determined by their simple orthospectrum. We also prove that two hyperbolic structures on the one-holed torus are isometric if and only if they have the same simple orthospectrum.

In the second part, we compute necessary and sufficient conditions on the Fenchel-Nielsen coordinates of a flute surface to be of the first kind. We show how this result can be used for other infinite-type surfaces as well. Then we give an application of this theorem to determine if a flute surface with twist parameters in $\{0, 1/2\}$ has ergodic geodesic flow.

Keywords: hyperbolic surfaces, orthospectrum, simple orthospectrum, infinite-type surfaces, first-kind surfaces, ergodic geodesic flow.

Résumé

Cette thèse est dédiée à l'étude de la géométrie hyperbolique sur des surfaces orientables. Tout au long de celle-ci, deux points de vue différents seront abordés. Premièrement, nous nous intéresserons aux surfaces de type fini avec bord à travers l'orthospectre et l'orthospectre simple. Puis, nous étudierons des surfaces de type infini à travers leur ensemble limite.

Dans la première partie, nous caractérisons la rigidité de l'orthospectre et de l'orthospectre simple : nous montrons que seulement un nombre fini de surfaces, à isométrie près, ont le même orthospectre. Pour les surfaces n'ayant qu'une seule composante de bord, nous montrons que cet énoncé est aussi vrai pour l'orthospectre simple. De plus, nous montrons que les surfaces sont génériquement déterminées par leur orthospectres et que les surfaces avec une seule composante de bord sont génériquement déterminées par leur orthospectre simple. Nous montrons également que deux structures hyperboliques sur un tore troué sont isométriques si et seulement si elles ont le même orthospectre simple.

Dans la seconde partie, nous calculons des conditions nécessaires et suffisantes sur les coordonnées de Fenchel-Nielsen d'une surface flûte pour qu'elle soit du premier type. Nous montrons comment ce résultat peut être utilisé pour d'autres surfaces de type infini. Puis, nous présentons une application de ce théorème pour déterminer si une surface flûte, dont les paramètres de twist appartiennent à $\{0, 1/2\}$, a un flot géodésique ergodique.

Mots-Clef: surfaces hyperboliques, orthospectre, orthospectre simple, surfaces de type infini, surfaces de premier type, flot géodésique ergodique.

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Chapter 1

Introduction

This thesis is about hyperbolic surfaces of finite-type and infinite-type. In this introduction, we will present some context around the study of hyperbolic surfaces. Then we present two different aspects of hyperbolic surfaces that are investigated in this thesis, as well as an overview of the different results we obtained and how they play a role in the wider context. All along this introduction we provide some historical context and notes sourced from [EH10], [AV20], the preface of [FN03] and the historical notes at the end of each chapter of [Rat19].

In 1907, the uniformization theorem was proved independently by Poincaré in Sur l'uniformisation des fonctions analytiques [Poi08] and by Koebe in Über die Uniformisierung der algebraischen Kurven [Koe10].

Theorem 1 (Uniformization theorem). *A simply connected Riemann surface is isomorphic to either the Riemann sphere $\mathbb{P}^1$, the complex plane $\mathbb{C}$, or the open unit disc $\mathbb{D} \approx \mathbb{H} \subset \mathbb{C}$.*

As the universal cover of a surface is simply connected, it has to be in one of the three categories described by the uniformization Theorem. More precisely, we have the following classification [Hub06, Theorem 1.8.8].

Theorem 2. *We have the following classification of surfaces by their universal cover.*

- *The Riemann sphere $\mathbb{P}^1$ is the universal cover only of itself.*
- *The plane $\mathbb{C}$ is the universal cover of itself, of the punctured plane $\mathbb{C} \setminus \{a\}$ and of all compact Riemann surfaces homeomorphic to a torus.*
- *All other Riemann surfaces have universal covering space isomorphic to $\mathbb{D}$.*

If the universal cover of a surface is $\mathbb{P}^1$, the metric of the surface is said to be spherical or elliptic, if it is $\mathbb{C}$ the metric is said to be Euclidean or parabolic and if it is $\mathbb{H}$ it is said to be hyperbolic. In other words, most surfaces admits a hyperbolic metric.

By the topological classification result (sometimes credited to Dehn and Heegard [DH07], sometimes to Brahana [Bra21]) we know that closed orientable surfaces are topologically determined by their genus. The classification can be extended to compact and non-compact surfaces and a general statement was given by Kerékjártó [Ker23] and Richard [Ric63]. Finite-type surfaces are topologically determined by their genus and number of boundary components and punctures, however,

more information is needed to topologically infinite type surfaces. Let S be a surface with genus g , n punctures and b boundary components (possibly infinitely many of each of them in the case of infinite-type surfaces) and define $\chi(S) = 2 - 2g - b - n$ its Euler characteristic. All surfaces with negative Euler characteristic fall in the third category of geometry and admit a hyperbolic structure.

There are different ways to define a space of hyperbolic surfaces, one that we will use for its practicality is Teichmüller space. Teichmüller space first appeared for closed orientable surfaces implicitly in Klein's 1883 paper *Neue Beiträge zur Riemannschen Functionentheorie* [Kle83] and in Poincaré's 1884 paper *Sur les groupes des équations linéaires* [Poi84]. Then it was explicitly introduced by Teichmüller in his 1939 paper *Extremale quasikonforme Abbildungen und quadratische Differentiale* [Tei40]. Although this space has historically been called "Teichmüller space", as Teichmüller was a convinced nazi, in the rest of this thesis we will call this space "Deformation space" instead and denote it as $\text{Def}(S)$.

To identify surfaces in the Deformation space we use different sets of coordinates. We use Fenchel-Nielsen coordinates, introduced by Fenchel and Nielsen in 1948 in *Discontinuous groups of isometries in the hyperbolic plane* [FN03], and Ushijima coordinates, introduced in 1999 for finite-type surfaces with boundary in *A canonical cellular decomposition of the Teichmüller space of compact surfaces with boundary* [Ush99].

The first objects studied in this thesis are the orthospectrum and simple orthospectrum. Both objects take inspiration into the length spectrum and the simple length spectrum which find their origin in a very simple question. In a famous paper [Kac66], Kac asked "Can one hear the shape of a drum?". The question can be formalized mathematically using the fact that the sound made by a drum is linked to the frequencies at which a drumhead vibrates, that is, the spectrum of the Laplacian. Kac's question then becomes: "Does the spectrum of the Laplacian of a planar domain determines the domain itself?". This question was then also asked for general Riemannian manifolds. In the case of closed hyperbolic surfaces, Huber [Hub61] and Selberg [Sel56] showed that the Laplacian spectrum determines and is determined by the length spectrum, which is the set of lengths of closed geodesics counted with multiplicities. This shifted the question to "Does the length spectrum of a hyperbolic surface determine the metric up to isometry?". In 1978, Vignéras provided the first example of isospectral non-isometric closed hyperbolic surfaces [Vig78] showing that the answer to the question is negative, and in 1985, Sunada gave a general criterion to construct isospectral non-isometric manifolds [Sun85]. On the other hand, McKean proved that there is a finite number of isospectral non-isometric hyperbolic surfaces [McK72]. In 1979, Wolpert showed that a generic hyperbolic surface is determined by its length spectrum [Wol79]. In 1985, Haas proved that the one-holed hyperbolic torus with a fixed boundary length is determined by the length spectrum [Haa85], and later Buser and Semmler removed the condition on the boundary length [BS88]. See [Bus10, Chapter 7.1] for more historical details on the subject. In the case of the simple spectrum (where only the simple closed geodesics are considered) the question is still open: there are no known examples of non-isometric hyperbolic surfaces with the same simple length spectrum. There is, however, a result by Baik, Choi and Kim showing that generic hyperbolic surfaces are also determined by their simple length spectrum [BCK20].

In 1993, Basmajian introduced the orthospectrum of hyperbolic surfaces with boundary, defined as the set of lengths of geodesic arcs orthogonal to the boundary (called orthogeodesics) counted with multiplicities [Bas93]. This object, analogous to the length spectrum, aims at taking the boundary of a surface more into account. In particular, Basmajian gave a formula to compute

the boundary length of a hyperbolic surface from its orthospectrum [Bas93]. Motivated by Kac's question, Masai and McShane considered the analogous problem for the orthospectrum: "Does the orthospectrum of a hyperbolic surface determine the metric up to isometry?". They showed that there is a finite number of non-isometric hyperbolic structures on a compact surface with a single boundary component sharing the same orthospectrum [MM23]. In the same paper, they showed that the one-holed torus is determined its orthospectrum. They also gave a general construction for isospectral non-isometric hyperbolic surfaces, thus answering the question in the negative.

In Chapter 3, we investigate further Masai and McShane's question, both for the orthospectrum and the simple orthospectrum, the simple orthospectrum being the multiset of lengths of simple orthogeodesics, counted with multiplicities. Note that even if the orthospectrum and simple orthospectrum are related, there is no known way to deduce one from the other. Denoting by $\mathcal{O}(X)$ and $\mathcal{O}_S(X)$ the orthospectrum and the simple orthospectrum of a hyperbolic surface X , our main results are the following.

First, we extend Masai and McShane's theorem to surfaces with any number of boundary components.

Theorem 3.2.2. *Let S_g^b be a compact, genus g surface of negative Euler characteristic with b boundary components. Let X be a hyperbolic structure on S_g^b with geodesic boundary. Then, up to isometry, there is a finite number of hyperbolic structures Y on S_g^b such that*

$$\mathcal{O}(Y) = \mathcal{O}(X).$$

We also prove the analogous result for the simple orthospectrum, in the case of surfaces with only one boundary component.

Theorem 3.2.3. *Let S_g^1 be a compact, genus g surface of negative Euler characteristic with one boundary components. Let X be a hyperbolic structure on S_g^1 with geodesic boundary. Then, up to isometry, there is a finite number of hyperbolic structures Y on S_g^1 such that*

$$\mathcal{O}_S(Y) = \mathcal{O}_S(X).$$

The proof of both theorems is separated into 2 steps: first we show that the set of surfaces Y such that $\mathcal{O}_S(Y) = \mathcal{O}_S(X)$ is compact, then that it is a discrete set. To prove the first step, we use very different techniques in the two cases. In particular, for Theorem 3.2.3 we use results of curves and arcs counting by Mirzakhani [Mir08] and Bell [Bel23] in opposition to the use of lengths of orthogeodesic that self-intersect.

We also establish an analogue of Wolpert's result both for the orthospectrum and the simple orthospectrum. In the case of the simple orthospectrum, we again restrict ourself to surface with only one boundary component.

Theorem 3.3.1. *Generic surfaces in $\text{Def}(S_g^b)$ are determined, up to isometry, by their orthospectrum. Similarly, generic surfaces in $\text{Def}(S_g^1)$ are determined, up to isometry, by their simple orthospectrum.*

We took inspiration in Wolpert's proof of the generic determination by the length spectrum: we show that the set of surfaces in the Deformation space such that there exist another non-isometric surface with the same orthospectrum (or simple orthospectrum) is included in a proper real analytic variety of $\text{Def}(S_g^b)$. One main difference with Wolpert's proof is the use of Ushijima coordinates

instead of Fenchel-Nielsen coordinates. Another key difference is the way we build open neighborhood in which this set is equal to a proper real analytic variety. This is also the main piece of the proof that differ for the orthospectrum and the simple orthospectrum.

Finally, we show an example of surfaces determined by their simple orthospectrum:

Theorem 3.4.2. *Let T and T' be two hyperbolic structures with geodesic boundary on the one-holed torus. Then T and T' are isometric if and only if $\mathcal{O}_S(T) = \mathcal{O}_S(T')$.*

The proof of this theorem is based on the use of Ushijima coordinates and the fact that on the one-holed torus the three shortest simple orthogeodesics are disjoint.

The second object of study of this thesis is a geometric property: being of the first kind.

Poincaré introduced the limit set of a discrete group of transformations of the unit disk in his 1882 paper *Mémoire sur les fonctions fuchsiennes* [Poi82]. For a hyperbolic surface $\mathbb{H}/\Gamma$ (with $\mathbb{H}$ the hyperbolic plane and Γ a discrete subgroup of orientation-preserving isometries of $\mathbb{H}$ without fixed points), we define the limit set $\Lambda(\Gamma)$ as the set of accumulation points in $\partial\mathbb{H}$ of orbits of Γ . By [Kat92] the limit set is either the whole boundary $\partial\mathbb{H}$, a point, two points or a nowhere dense subset of $\partial\mathbb{H}$. Hyperbolic surfaces whose fuchsian groups have limit set equal to $\partial\mathbb{H}$ are said to be of the first kind, otherwise, they are of the second kind.

For finite-type surfaces it is easy to distinguish first kind surfaces from second kind surfaces: a surface is of the first kind if and only if it has no boundary components or funnels. But for infinite-type surfaces, the fixed topology of a surface with a distinction between puncture and boundary components is not enough to determine if a surface is of the first kind. Basmajian and Šarić developed the concept of visible end in 2019 to study first kind surfaces: they proved that a hyperbolic surface is of the first kind if and only if none of its end is visible [BS19].

Knowing if a surface is of the first kind is also useful as it is a prerequisite to the absence of a Green's function on the surface, which is a property with many equivalent characterization that reflect as many facet and point of views of surfaces.

Theorem 3. *Let X be a Riemann surface, X the following properties are equivalent:*

- *There is no Green's function on X [AS60],*
- *the boundary at infinity has zero harmonic measure [AS60],*
- *the Poincaré series $\sum_{\gamma \in \Gamma} e^{-d(z, \gamma(z))}$ is divergent, where $X = \mathbb{H}/\Gamma$, $d(.,.)$ is the hyperbolic distance in $\mathbb{H}$ and $z \in \mathbb{H}$ [Tsu75, Gar86],*
- *the Brownian motion on X is recurrent [FM01],*
- *the geodesic flow on the unit tangent bundle $T^1(X)$ of X is ergodic [Nic89],*
- *the limit set of a quasiconformal deformation of Γ has Bowen's property [Bis01, AZ90], and*
- *almost every horizontal trajectory of every finite area holomorphic quadratic differential on X is recurrent [vS24].*

These properties on Riemann surfaces have been well studied (as in [AS60, Mil77] for example) from a complex analytical point of view. A more recent approach, however, is to study these properties from a hyperbolic geometry point of view, and in particular through Fenchel-Nielsen coordinates.

Basmajian, Hakobyan and Šarić gave in 2022 several sufficient criteria on Fenchel-Nielsen coordinates to tell if a surface has an ergodic geodesic flow. They start from a general case for infinite-type surfaces and refine their results to flute surfaces, Loch Ness surfaces, Jacob's ladder surfaces, Cantor tree surfaces and Abelian covers of compact surfaces [BHŠ22]. However, they only have an equivalence for flute surfaces with twist parameters equal to 0 (and flute surfaces with twist parameters equal to $1/2$ and some other additional conditions). Conversely, McMullen [McM98] showed that, for the Cantor tree, if the lengths of all geodesics in a pants decomposition are bounded away from zero and infinity, then the surface do not have an ergodic geodesic flow. In 2023, Pandazis and Šarić extend these results with necessary and sufficient conditions in the case of flute surfaces with twist parameters equal to $1/2$ and for Loch Ness surfaces with some additional conditions on its Fenchel-Nielsen coordinates [PŠ23]. In some cases, described by Pandazis and Šarić as "symmetric" surfaces, properties of Theorem 3 are even equivalent to being of the first kind [PŠ23].

The work in Chapter 4 is a result of a collaboration with Gordillo Herrerías where we look at the case of flute surfaces [GL25]. Basmajian, Hakobyan and Šarić [BHŠ22] and later Pandazis and Šarić in [PŠ23] gave criteria with additional condition on the Fenchel-Nielsen coordinates to determinate if a flute surface is of the first kind. In our main result we lift these additional conditions on the Fenchel-Nielsen coordinates to give a criterion to determine if a flute surface is of the first kind.

Theorem 4.2.3. *Let X be a tight flute surface of Fenchel-Nielsen coordinates $\{\ell_n, t_n\}_{n \in \mathbb{N}^*}$. There exist an infinite family $\mathcal{U}$ of explicit sequences that only depends on twists such that X is of the first kind if and only if, for all sequences $(u_n)_{n \in \mathbb{N}^*} \in \mathcal{U}$, we have*

$$\sum_{n=1}^{\infty} \left(e^{-\frac{\ell_{n+1}}{2}} + e^{-\frac{\ell_n}{2}} \right) \cosh(u_n \ell_n + \dots + u_1 \ell_1) = \infty.$$

The proof of this theorem uses the concept of visible end, together with hyperbolic trigonometric estimates. Then we explore consequences and applications of this theorem. In particular, we look at cases for which to be of the first kind is equivalent to having an ergodic geodesic flow.

Theorem 4.3.6. *Let X be a tight flute surface of Fenchel-Nielsen coordinates $\{\ell_n, t_n\}_{n \in \mathbb{N}^*}$ such that for all n we have $t_n \in \{0, \frac{1}{2}\}$. There exist explicit sequences $(u_n)_{n \in \mathbb{N}^*}$ and $(v_n)_{n \in \mathbb{N}^*}$ such that X has an ergodic geodesic flow if and only if*

$$\sum_{n=1}^{\infty} \left(e^{-\frac{\ell_{n+1}}{2}} + e^{-\frac{\ell_n}{2}} \right) \cosh(u_n \ell_n + \dots + u_1 \ell_1) = \infty.$$

Moreover, if X has an increasing sequence of cuff lengths ℓ_n , then X has an ergodic geodesic flow if and only if

$$\sum_{n=1}^{\infty} \left(e^{-\frac{\ell_{n+1}}{2}} + e^{-\frac{\ell_n}{2}} \right) e^{-v_{2n+1}(u_n \ell_n + \dots + u_1 \ell_1)} = \infty.$$

The second sum in this theorem allows us to recover Basmajian, Hakobyan and Šarić's result when $t_n = 0$ for all n , and Pandazis and Šarić's result when $t_n = 1/2$ for all n . In particular, this theorem completes Hakobyan, Pandazis and Šarić [HPS24, Table 1] that summarized the known results for flute surfaces with twists $t_n \in \{0, 1/2\}$.

Structure of the thesis

This thesis is divided in three chapters.

Chapter 2 is dedicated to preparations and definitions relative to hyperbolic surfaces in Section 2.1, to the Deformation space in Section 2.2 and to various hyperbolic geometry and trigonometric relations that will be useful all along the thesis in Section 2.3.

Chapter 3 focuses on the orthospectrum and simple orthospectrum. We recall the notion of orthospectrum and simple orthospectrum, their properties and properties of some subspaces of surfaces in Section 3.1. In Section 3.2, we state and prove Theorem 3.2.2 and Theorem 3.2.3. In Section 3.3, we state and prove Theorem 3.3.1. In Section 3.4, we show Theorem 3.4.2, which gives an example of surface determined up to isometry by its simple orthospectrum.

Finally, in Chapter 4, we focus on first kind flute surfaces. We recall the concept of visible end in Section 4.1. In Section 4.2, we give a criterion on the Fenchel-Nielsen coordinates to determine if a flute surface is of the first kind and prove Theorem 4.2.3. In Section 4.3 we explore an application of this criterion by proving Theorem 4.3.6 to determine if a flute surface has an ergodic geodesic flow.

Introduction

Cette thèse porte sur les surfaces hyperboliques de type fini et de type infini. Dans cette introduction, nous allons présenter une partie du contexte autour de l'étude des surfaces hyperboliques. Ensuite nous présenterons deux aspects différents des surfaces hyperboliques qui sont abordés dans cette thèse, ainsi qu'une vue globale des différents résultats que nous obtenons et leur rôle dans leur contexte respectif. Tout au long de cette introduction nous partagerons un peu de contexte historique provenant de [EH10], [AV20], la préface de [FN03] et des notes historiques à la fin de chaque chapitre de [Rat19].

En 1907, le théorème d'uniformisation a été prouvé indépendamment par Poincaré dans Sur l'uniformisation des fonctions analytiques [Poi08] et par Koebe dans Über die Uniformisierung der algebraischen Kurven [Koe10].

Théorème 1 (Théorème d'uniformisation). *Une surface de Riemann simplement connexe est isomorphe soit à la sphère de Riemann $\mathbb{P}^1$, soit au plan complexe $\mathbb{C}$, soit au disque unité ouvert $\mathbb{D} \approx \mathbb{H} \subset \mathbb{C}$.*

Comme le revêtement universel d'une surface est simplement connexe, il doit appartenir l'une de ces trois catégories décrites par le théorème d'uniformisation. Plus précisément, nous avons la classification suivante [Hub06, Theorem 1.8.8].

Théorème 2. *Nous avons la classification suivante des surfaces par leur revêtement universel.*

- *La sphère de Riemann $\mathbb{P}^1$ n'est que le revêtement universel d'elle-même.*
- *Le plan $\mathbb{C}$ est le revêtement universel de lui-même, du plan avec une pointe $\mathbb{C} \setminus \{a\}$ et de toutes les surfaces de Riemann compacte homéomorphe à un tore.*
- *Toutes les autres surfaces de Riemann ont un revêtement universel isomorphe à $\mathbb{D}$.*

Si le revêtement universel d'une surface est $\mathbb{P}^1$, sa métrique est dite sphérique ou elliptique, si c'est $\mathbb{C}$ sa métrique est dite Euclidienne ou parabolique et si c'est $\mathbb{H}$ elle est dite hyperbolique. Autrement dit, la plupart des surfaces admettent une métrique hyperbolique.

Par le résultat de classification topologique (parfois crédité à Dehn et Heegard [DH07], parfois à Brahana [Bra21]) nous savons que les surfaces orientables fermées sont topologiquement déterminées par leur genre. La classification peut être étendue aux surfaces compactes et non-compactes et un énoncé général a été donné par Kerékjártó [Ker23] et Richard [Ric63]. Les surfaces de type fini sont topologiquement déterminées par leur genre, leur nombre de composantes de bord et de pointes, cependant, plus d'information est nécessaire pour les surfaces de type infini. Soit S

une surface de genre g , avec n pointes et b composantes de bord (avec, pour chaque valeur, possiblement un nombre infini dans le cas des surfaces de type infini), définissons $\chi(S) = 2 - 2g - b - n$ sa caractéristique d'Euler. Toutes les surfaces de caractéristique d'Euler négative font partie de la troisième catégorie géométrique et admettent une structure hyperbolique.

Il y a différentes manières de définir des espaces de surfaces hyperboliques, un espace que nous utiliserons pour son côté pratique est l'espace de Teichmüller. L'espace de Teichmüller apparaît pour la première fois implicitement pour les surfaces orientables fermées dans l'article de Klein de 1883 *Neue Beiträge zur Riemannschen Functionentheorie* [Kle83] et dans l'article de Poincaré de 1884 *Sur les groupes des équations linéaires* [Poi84]. Ensuite, il a été explicitement introduit par Teichmüller dans son article de 1939 *Extremale quasikonforme Abbildungen und quadratische Differential* [Tei40]. Bien que cet espace est historiquement appelé "espace de Teichmüller", comme Teichmüller était un nazi convaincu, dans le reste de ce manuscrit nous l'appellerons "espace de Déformations" et nous le noterons $\text{Def}(S)$.

Pour identifier les surfaces dans l'espace de Déformation nous utilisons différents ensembles de coordonnées. Nous utilisons les coordonnées de Fenchel-Nielsen, introduit par Fenchel et Nielsen en 1948 dans *Discontinuous groups of isometries in the hyperbolic plane* [FN03], et les coordonnées de Ushijima, introduit en 1999 pour les surfaces de type fini avec du bord dans *A canonical cellular decomposition of the Teichmüller space of compact surfaces with boundary* [Ush99].

Les premiers objets d'étude de cette thèse sont l'orthospectre et l'orthospectre simple. Ces deux objets puisent leur inspiration dans le spectre des longueurs et le spectre simple des longueurs dont l'origine provient d'une question très simple. Dans un article maintenant populaire [Kac66], Kac demande "Peut-on entendre la forme d'un tambour ?". Cette question a été formalisée mathématiquement en utilisant le fait que le son produit par un tambour est lié aux fréquences auxquelles un tambour vibre, c'est à dire, au spectre du Laplacien. La question de Kac devient alors : "Est ce que le spectre du Laplacien d'un domaine plan détermine le domaine ?". Cette question a ensuite été posée pour les variétés Riemanniennes en général. Dans le cas des surfaces hyperboliques fermées, Huber [Hub61] et Selberg [Sel56] montrent que le spectre du Laplacien détermine et est déterminé par le spectre des longueurs, qui est le multi-ensemble des longueurs des géodésiques fermées comptées avec multiplicité. Cela a transformé la question en "Est ce que le spectre des longueurs d'une surface hyperbolique détermine sa métrique à isométrie près ?". En 1978, Vignéras produit les premiers exemples de surfaces hyperboliques fermées non-isométriques entre elles et isospectrales [Vig78], montrant ainsi que la réponse à cette question est négative, et en 1985, Sunada donne un critère général de construction de variétés non-isométriques et isospectrales [Sun85]. D'un autre côté, McKean prouve qu'il y a un nombre fini de surfaces hyperboliques non-isométriques de même spectre des longueurs [McK72]. En 1979, Wolpert montre qu'une surface hyperbolique générique est déterminée par son spectre des longueurs [Wol79]. En 1985, Haas prouve que le tore à un trou avec une longueur de bord fixé est déterminé par son spectre des longueurs [Haa85], et plus tard Buser et Semmler retire la condition sur la longueur du bord [BS88]. Voir [Bus10, Chapter 7.1] pour plus de détail historique sur le sujet. Dans le cas du spectre simple des longueurs (où seule les longueurs des géodésiques fermées simples sont considérées) la question est encore ouverte : il n'y a pas d'exemple connu de surfaces hyperboliques non-isométriques ayant le même spectre simple des longueurs. Il y a cependant, un résultat de Baik, Choi et Kim montrant qu'une surface hyperbolique générique est aussi déterminée par son spectre simple des longueurs [BCK20].

En 1993, Basmajian introduit l'orthospectre des surfaces hyperboliques à bord, définis comme le

multi-ensemble des longueurs d'arc géodésiques orthogonaux aux bord (appelées orthogéodésiques) comptés avec multiplicité [Bas93]. Cet objet, analogue au spectre des longueurs, vise à prendre plus en compte le bord d'une surface. En particulier, Basmajian prouve une formule qui permet de calculer la longueur du bord d'une surface hyperbolique à partir de son orthospectre [Bas93]. Motivés par la question de Kac, Masai et McShane considère un problème analogue pour l'orthospectre : "Est ce que l'orthospectre d'une surface hyperbolique détermine sa métrique à isométrie près ?". Ils montrent qu'il y a un nombre fini de structures hyperboliques non-isométriques sur une surface compacte à une seule composante de bord ayant le même orthospectre [MM23]. Dans le même article, ils montrent qu'un tore à un trou est déterminé par son orthospectre. Ils donnent aussi une construction général de surfaces hyperboliques non-isométriques de même orthospectre, répondant ainsi à la question de façon négative.

Dans le Chapitre 3, nous poussons l'exploration de la question de Masai et McShane, à la fois pour l'orthospectre et l'orthospectre simple, l'orthospectre simple étant le multi-ensemble des longueurs des orthogéodésiques simples, comptées avec multiplicité. Notons que même si l'orthospectre et l'orthospectre simple sont liés, il n'y a pas de manière connue de déduire l'un de l'autre. Notons $\mathcal{O}(X)$ et $\mathcal{O}_S(X)$ l'orthospectre et l'orthospectre simple d'une surface hyperbolique X , nos résultats principaux sont les suivants.

Premièrement, nous étendons le théorème de Masai et McShane aux surfaces ayant n'importe quel nombre de composantes de bord.

Théorème 3.2.2. *Soit S_g^b une surface compacte, de genre g , avec b composantes de bord et de caractéristique d'Euler négative. Soit X une structure hyperbolique sur S_g^b de bord géodésique. Alors, à isométrie près, il y a un nombre fini de structures hyperboliques Y sur S_g^b telles que*

$$\mathcal{O}(Y) = \mathcal{O}(X).$$

Nous montrons aussi un résultat analogue pour l'orthospectre simple, dans le cas des surfaces avec une seule composantes de bord.

Théorème 3.2.3. *Soit S_g^1 une surface compacte, de genre g , avec une seule composante de bord et de caractéristique d'Euler négative. Soit X une structure hyperbolique sur S_g^1 de bord géodésique. Alors, à isométrie près, il y a un nombre fini de structures hyperboliques Y sur S_g^1 telles que*

$$\mathcal{O}_S(Y) = \mathcal{O}_S(X).$$

Pour les deux théorèmes la preuve est séparée en deux étapes : D'abord nous montrons que l'ensemble des surfaces Y telles que $\mathcal{O}_S(Y) = \mathcal{O}_S(X)$ est compact, puis que c'est un ensemble discret. Pour montrer cette première étape, nous utilisons des techniques très différentes dans chacun des cas. En particulier, pour le Théorème 3.2.3 nous utilisons des résultats de comptage de courbes et d'arcs de Mirzakhani [Mir08] et de Bell [Bel23], en opposition à l'usage de longueurs d'orthogéodésiques qui s'auto-intersectent.

Nous établissons également un résultat analogue à celui de Wolpert à la fois pour l'orthospectre et pour l'orthospectre simple. Dans le cas de l'orthospectre simple, nous nous restreignons à nouveau aux surfaces ayant une seule composante de bord.

Théorème 3.3.1. *Les surfaces génériques dans $\text{Def}(S_g^b)$ sont déterminées, à isométrie près, par leur orthospectre. Similairement, les surfaces génériques dans $\text{Def}(S_g^1)$ sont déterminées, à isométrie près, par leur orthospectre simple.*

Nous prenons inspiration dans la preuve de Wolpert de la détermination générique des surfaces hyperboliques par le spectre des longueurs: nous montrons que l'ensemble des surfaces dans l'espace de Déformation telles qu'il en existe une autre non-isométrique avec le même orthospectre (ou orthospectre simple) est inclut dans une variété analytique réelle locale propre de $\text{Def}(S_g^b)$. La différence principale avec la preuve de Wolpert est l'usage des coordonnées de Ushijima à la place des coordonnées de Fenchel-Nielsen. Une autre différence clef est la manière dont nous construisons un voisinage ouvert à l'intérieur duquel l'ensemble sera égal à une variété analytique réelle propre. C'est aussi là que se trouve la différence principale entre la preuve pour l'orthospectre et la preuve pour l'orthospectre simple.

Enfin, nous montrons un exemple de surfaces déterminées par leur orthospectre simple:

Théorème 3.4.2. *Soit T et T' deux structures hyperboliques de bord géodésique sur le tore à un trou. Alors T et T' sont isométriques si et seulement si $\mathcal{O}_S(T) = \mathcal{O}_S(T')$.*

Cette preuve se base sur l'usage des coordonnées de Ushijima et le fait que sur le tore à un trou, les trois orthogéodésiques simples les plus courtes sont disjointes.

Le second objet d'étude de cette thèse est une propriété géométrique: être de première espèce.

Poincaré introduit l'ensemble limite d'un groupe discret de transformations du disque unité dans son article de 1882 Mémoire sur les fonctions fuchsiennes [Poi82]. Pour une surface hyperbolique $\mathbb{H}/\Gamma$ (avec $\mathbb{H}$ le plan hyperbolique et Γ un sous-groupe discret d'isométries préservant l'orientation de $\mathbb{H}$ et sans points fixes), nous définissons l'ensemble limite $\Lambda(\Gamma)$ comme l'ensemble des points d'accumulations dans $\partial\mathbb{H}$ des orbites de Γ . Par [Kat92] l'ensemble limite est soit tout le bord $\partial\mathbb{H}$, soit un point, soit deux points, soit un sous-ensemble nulle part dense de $\partial\mathbb{H}$. Les surfaces hyperboliques dont le groupe Γ a un ensemble limite égale à $\partial\mathbb{H}$ sont dites de première espèce, autrement, elles sont dites de seconde espèce.

Pour les surfaces de type fini il est facile de distinguer les surfaces de première espèce des surfaces de seconde espèce : une surface est de première espèce si et seulement si elle n'a pas de composantes de bord ou de vasque. Mais pour les surfaces de type infini, fixer la topologie de la surface en distinguant pointes et composantes de bord n'est pas suffisant pour déterminer si une surface est de première espèce. Basmajian et Šarić développent le concept de bouts visibles en 2019 pour étudier les surfaces de première espèce : ils prouvent qu'une surface hyperbolique est de première espèce si et seulement si aucun de ses bouts n'est visibles [BŠ19].

Savoir qu'une surface est de première espèce est aussi utile puisque c'est un prérequis pour l'absence de fonction de Green sur la surface, une propriété avec de nombreuses caractérisations équivalentes reflétant autant de facettes et de points de vue sur les surfaces.

Théorème 3. *Soit $X = \mathbb{H}/\Gamma$ une surface de Riemann, X les propriétés suivantes sont équivalentes :*

- *Il n'y a pas de fonction de Green sur X [AS60],*
- *le bord à l'infini est de mesure harmonique nulle [AS60],*
- *la série de Poincaré $\sum_{\gamma \in \Gamma} e^{-d(z, \gamma(z))}$ est divergente, où $d(.,.)$ est la distance hyperbolique dans $\mathbb{H}$ et $z \in \mathbb{H}$ [Tsu75, Gar86],*
- *le mouvement Brownien sur X est récurrent [FM01],*
- *le flot géodésique sur le fibré tangent unitaire $T^1(X)$ de X est ergodique [Nic89],*

- l'ensemble limite des déformations quasi-conforme de Γ a la propriété de Bowen [Bis01, AZ90], et
- presque toute trajectoire horizontal de chaque différentiel quadratique holomorphe d'aire fini sur X est récurrente [vS24].

Ces propriétés sur les surfaces de Riemann ont été étudié en profondeur (comme dans [AS60, Mil77] par exemple) d'un point de vue d'analyse complexe. Une approche plus récente, cependant, est celle d'étudier ces propriétés d'un point de vue de la géométrie hyperbolique, et en particulier, à travers les coordonnées de Fenchel-Nielsen. Basmajian, Hakobyan et Šarić donnent en 2022 plusieurs critères suffisant sur les coordonnées de Fenchel-Nielsen pour savoir si une surface à un flot géodésique ergodique. Ils commencent par un critère général pour les surfaces de type infini puis précise leur résultat pour les surfaces flûtes, les surfaces Loch Ness, les surfaces échelles de Jacob, les surfaces arbres de Cantor et pour les revêtements abéliens de surfaces compactes [BHŠ22]. Cependant, ils ne montrent une équivalence que pour les surfaces flûtes dont les paramètres de twist sont 0 (et les surfaces flûte dont les paramètres de twist sont $1/2$ et avec d'autres conditions additionnelles). D'un autre côté, McMullen [McM98] montre que, pour une surface arbre de Cantor, si les longueurs des géodésiques d'une décomposition en pantalons sont minorées et majorées par des constantes finies et strictement plus grandes que zéro, alors la surface n'admet pas de flot géodésique ergodique. En 2023, Pandazis et Šarić étendent ces résultats avec des conditions nécessaires et suffisantes dans le cas des surfaces flûtes de paramètres de twist $1/2$ et pour les surfaces Loch Ness avec des conditions additionnelles sur les coordonnées de Fenchel-Nielsen [PŠ23]. Dans certains cas, décrits par Pandazis et Šarić comme des surfaces "symétriques", les propriétés du Théorème 3 sont même équivalentes au fait d'être de première espèce [PŠ23].

Le travail dans le Chapitre 4 est le résultat d'une collaboration avec Gordillo Herrerías où nous nous penchons sur le cas des surfaces flûtes [GL25]. Basmajian, Hakobyan et Šarić [BHŠ22] et plus tard Pandazis and Šarić 2023 [PŠ23] ont donné des critères avec des conditions additionnelles sur les coordonnées de Fenchel-Nielsen pour déterminer si une surface flûte est de première espèce. Dans le résultat principal de ce chapitre nous retirons ces conditions additionnelles sur les coordonnées de Fenchel-Nielsen pour donner un critère pour déterminer si une surface flûte est de première espèce.

Théorème 4.2.3. *Soit X une surface flûte serrée de coordonnées de Fenchel-Nielsen $\{\ell_n, t_n\}_{n \in \mathbb{N}^*}$. Il existe une famille infini $\mathcal{U}$ de suites explicites qui ne dépendent que des paramètres de twist telle que X est de première espèce si et seulement si, pour toutes suites $(u_n)_{n \in \mathbb{N}^*} \in \mathcal{U}$, nous avons*

$$\sum_{n=1}^{\infty} \left(e^{-\frac{\ell_{n+1}}{2}} + e^{-\frac{\ell_n}{2}} \right) \cosh(u_n \ell_n + \dots + u_1 \ell_1) = \infty.$$

La preuve de ce théorème utilise le concept de bouts visibles avec des estimés de trigonométrie hyperbolique. Ensuite, nous explorons les conséquences et applications de ce théorème. En particulier, nous nous intéressons aux cas pour lesquels être de première espèce est équivalent avec le fait d'avoir un flot géodésique ergodique.

Théorème 4.3.6. *Soit X une surface flûte serrée de coordonnées de Fenchel-Nielsen $\{\ell_n, t_n\}_{n \in \mathbb{N}^*}$ telle que pour tout n nous avons $t_n \in \{0, \frac{1}{2}\}$. Il existe deux suites explicites $(u_n)_{n \in \mathbb{N}^*}$ et $(v_n)_{n \in \mathbb{N}^*}$*

telles que X a un flot géodésique ergodique si et seulement si

$$\sum_{n=1}^{\infty} \left(e^{-\frac{\ell_{n+1}}{2}} + e^{-\frac{\ell_n}{2}} \right) \cosh(u_n \ell_n + \dots + u_1 \ell_1) = \infty.$$

De plus, si X a une suite de paramètre de longueurs ℓ_n croissante, alors X a un flot géodesique ergodique si et seulement si

$$\sum_{n=1}^{\infty} \left(e^{-\frac{\ell_{n+1}}{2}} + e^{-\frac{\ell_n}{2}} \right) e^{-v_{2n+1}(u_n \ell_n + \dots + u_1 \ell_1)} = \infty.$$

La seconde somme de ce théorème permet de retrouver les résultats de Basmajian, Hakobyan et Šarić lorsque $t_n = 0$ pour tout n , et de Pandazis et Šarić lorsque $t_n = 1/2$ pour tout n . En particulier, ce théorème complète la [HPS24, Table 1] de Hakobyan, Pandazis et Šarić qui résume les résultats connus pour les surfaces flûtes de paramètres de twist dans $\{0, 1/2\}$.

Structure de la thèse

Cette thèse est divisée en trois chapitres.

Le Chapitre 2 est dédié aux prérequis et aux définitions relatives aux surfaces hyperboliques dans la Section 2.1, à l'espace de Déformation dans la Section 2.2 et à diverses relations de géométrie et trigonométrie hyperbolique dans la Section 2.3, qui seront utiles tout au long de ce manuscrit.

Le Chapitre 3 se concentre sur l'orthospectre et l'orthospectre simple. Nous rappelons les notions d'orthospectre et d'orthospectre simple, ainsi que leur propriétés et les propriétés de sous-espace de surfaces dans la Section 3.1. Dans la Section 3.2, nous énonçons et prouvons le Théorème 3.2.2 et le Théorème 3.2.3. Dans la Section 3.3, nous énonçons et prouvons le Théorème 3.3.1. Dans la Section 3.4, nous montrons le Théorème 3.4.2, qui nous donne un exemple de surface déterminée, à isométrie près, par son orthospectre simple.

Finalement, dans le Chapitre 4, nous nous concentrons sur les surfaces flûtes de première espèce. Nous rappelons le concept de bouts visibles dans la Section 4.1. Dans la Section 4.2, nous donnons un critère sur les coordonnées de Fenchel-Nielsen pour déterminer si une surface flûte est de première espèce et nous prouvons le Théorème 4.2.3. Dans la Section 4.3 nous explorons une application de ce critère en prouvant le Théorème 4.3.6 qui détermine si une surface flûte à un flot géodésique ergodique.

Chapter 2

Context

2.1 Hyperbolic surfaces

Let $\mathbb{H}$ be the hyperbolic plane, its group of orientation-preserving isometries is homeomorphic to $\mathrm{PSL}(2, \mathbb{R})$ [Kat92].

Definition 2.1.1. A Fuchsian group Γ is a discrete subgroup of the group of orientation-preserving isometries of $\mathbb{H}$.

An important space associated to a Fuchsian groups is its limit set.

Definition 2.1.2. The limit set $\Lambda(\Gamma)$ of Γ is the set of accumulation points in $\partial\mathbb{H}$ of orbits of Γ .

See [Kat92] for more details on Fuchsian groups. Let us define hyperbolic surfaces, which will be the central objects of this work.

Definition 2.1.3. A hyperbolic surface X is the quotient of $\mathbb{H}$ by a Fuchsian group Γ without fixed points or the quotient of the convex hull of the limit set of a Fuchsian group Γ without fixed points by Γ . We say that Γ is the Fuchsian group of X .

In particular, the fundamental group of a hyperbolic surface can be identified with its Fuchsian group, and its universal cover to a convex subset of $\mathbb{H}$.

On one hand, properties of the Fuchsian group of a surface translate to properties on the surface. One of them comes from the following result [Kat92, Theorem 3.4.6].

Proposition 2.1.4. *Let Γ be a Fuchsian group. If its limit set $\Lambda(\Gamma)$ contains more than two points then either*

- $\Lambda(\Gamma) = \partial\mathbb{H}$ or,
- $\Lambda(\Gamma)$ is a perfect nowhere dense subset of $\partial\mathbb{H}$.

This dichotomy leads to the following definition.

Definition 2.1.5. A Fuchsian group Γ is of the first kind if $\Lambda(\Gamma) = \partial\mathbb{H}$. Otherwise, it is of the second kind.

We say that a surface X is of the first kind (respectively second kind) if its Fuchsian group is of the first kind (respectively second kind).

On the other hand, hyperbolic surfaces inherit the metric of the hyperbolic plane and thus are locally isometric to the hyperbolic plane or half of the hyperbolic plane. Throughout the thesis, we will denote hyperbolic surfaces as X or Y .

Definition 2.1.6. A surface is a finite-type surface if its Fuchsian group is finitely generated, otherwise it is an infinite-type surface.

We always assume that surfaces are orientable and we will denote topological surfaces as S . Finite-type surfaces can be described topologically with their genus g , number of boundary components b and number of punctures n . We denote topological finite-type surfaces by $S_g^{b,n}$ or S_g^b when there are no punctures on the surface. A recurring finite-type surface will be a pair of pants, a surface homeomorphic to $S_0^{n,b}$ such that $n + b = 3$.

Definition 2.1.7. The Euler characteristic of $S_g^{b,n}$ is $\chi(S_g^{b,n}) = 2 - 2g - b - n$.

We allow the Euler characteristic to be infinite and extend its definition to infinite-type surfaces: $\chi(S) = 2 - 2g - b - n$ with g , b and n possibly infinite and b the number of boundary components and funnels. Thus $\chi(S)$ can be equal to $-\infty$.

Genus, number of boundary components and number of punctures are necessary but not enough to topologically describe infinite-type surfaces. On top of that, we need the notion of ends and space of ends. Let S be a topological surface. An *exiting sequence* is a sequence $\{U_n\}_{n \in \mathbb{N}}$ of connected open subsets of S satisfying:

1. the sequence is nested: $U_n \subset U_m$ whenever $n > m$,
2. U_n is not relatively compact for any $n \in \mathbb{N}$,
3. U_n has compact boundary for all $n \in \mathbb{N}$, and
4. any relatively compact subset of X intersects only finitely many of the U_n 's.

Two exiting sequences $\{U_n\}$ and $\{V_n\}$ are *equivalent* if for every $n \in \mathbb{N}$, there exists $m \in \mathbb{N}$ such that $U_m \subset V_n$ and $V_m \subset U_n$.

Definition 2.1.8. An *end* e of S is an equivalence class of exiting sequences, and the collection of all ends forms the *space of ends* $E(X)$.

The *space of ends* $E(S)$ of S is endowed with the following topology. For an open set $U \subset S$, define

$$U^* = \{[\{U_n\}_{n \in \mathbb{N}}] \in E(S) \mid U_n \subset U \text{ for } n \text{ large enough}\}.$$

The collection

$$\mathcal{B} = \{U^* \mid U \subset S \text{ is open with compact boundary}\}$$

forms a topological basis of $E(S)$. If U is an open set with compact boundary and $e \in U^*$, then we abuse notation and say that U is a neighborhood of the end e . With this topology, the space $E(X)$ is compact and totally disconnected. We can consider the union topology in $X \cup E(X)$.

There are also different ways to topologically describe ends:

Definition 2.1.9. An end is *planar* if it admits a neighborhood that is embeddable in the plane. Otherwise the end is *non-planar*.

Definition 2.1.10. An end is a *finite-type end* if it is planar and isolated in the space of ends. Otherwise, it is an infinite-type end.

In particular, we remark that finite-type surfaces can only have finite-type ends (and no infinite-type ends).

Let $E_{\text{np}}(S)$ be the space of non-planar ends, Richards establishes the classification of infinite-type surfaces [Ric63] as follows.

Theorem 2.1.11. Two orientable surfaces S and S' are topologically equivalent if and only if they have the same genus and the pair $(E(S), E_{\text{np}}(S))$ is homeomorphic to the pair $(E(S'), E_{\text{np}}(S'))$.

For a broader overview and further discussion, we refer the reader to [AV20].

Now that we can topologically describe hyperbolic surfaces, let us define different ways to decompose them into smaller pieces using curves and arcs.

Definition 2.1.12. A closed curve on S is *essential* if it is not homotopic to a boundary component or to a puncture. It is *primitive* if it is not homotopic to any proper power of another curve. It is *simple* if it has no self-intersection.

We will always assume closed curves to be primitive. With these descriptors of curves we can define multi-curves on finite-type surfaces:

Definition 2.1.13. A *multi-curve* is a finite formal sum $\gamma = \sum_i \alpha_i \gamma_i$ of pairwise disjoint, non-homotopic, essential, primitive, simple closed curves γ_i with positive coefficients α_i known as weights. It is *integral* if the weights are integers and *rational* if the weights are rational numbers. The set of integral multi-curves on $S_g^{b,n}$ is denoted

$$\mathcal{ML}_{g,b+n}(\mathbb{Z}).$$

Note that a simple closed curve can be seen as an integral multi-curve with only one weight that is equal to 1. This notation of the set of integral multi-curves comes from the notation of the set of laminations that contains it. See [FLP12] and [ES22] for more details on laminations.

A path is said to be *geodesic* (or a geodesic) if it locally minimizes the distance. Closed geodesic curves (or just closed geodesics) have the following properties [Bus10, Theorem 1.6.6].

Theorem 2.1.14. A closed geodesic of a hyperbolic surface is the unique shortest geodesic representative in the homotopy class of an essential closed curve.

Definition 2.1.15. The *systole* $\text{sys}(X)$ of a hyperbolic surface X is the shortest length of an essential closed curve on X .

Note that the systole is realized by the length of a simple closed geodesic, unless X is a pair of pants.

Definition 2.1.16. Let S be a surface of negative Euler characteristic. A *pants decomposition* of S is a maximal collection of pairwise non-homotopic, disjoint, essential simple closed curves on S and such that any compact set on the surface is intersected by only finitely many elements of the pants decomposition.

Remark 2.1.17. For a compact surface $S_g^{b,n}$, the cardinality of a pants decomposition is $3g + b + n - 3$ and a pants decomposition cuts $S_g^{b,n}$ into $2g + b + n - 2$ pairs of pants [Abi80].

When we study finite-type surfaces with boundary, it can be useful to change the perspective from the closed curve/closed geodesic point of view to the arc/orthogeodesic point of view.

Definition 2.1.18. An arc on S_g^b is *essential* if it is not homotopic, relatively to the boundary, into a boundary component. It is *simple* if it has no self-intersections.

Definition 2.1.19. An *orthogeodesic* on a hyperbolic surface X is a geodesic arc orthogonal to the boundary of X .

In particular, orthogeodesics have the following properties shown in [Bus10, Theorem 1.5.3].

Theorem 2.1.20. *An orthogeodesic is the shortest geodesic representative of the homotopy class relative to the boundary of an essential arc. Moreover, it is the unique geodesic orthogonal to the boundary in its homotopy class.*

Similarly to the systole, we also define the orthosystole.

Definition 2.1.21. The *orthosystole* $\text{osys}(X)$ of a hyperbolic surface X is the shortest length of an orthogeodesic on X .

Definition 2.1.22. Let X be a hyperbolic surface and let $\gamma \subset X$ be a closed geodesic. An *orthoray* based at γ is a geodesic ray $\rho : [0, \infty) \rightarrow X$ orthogonal to γ at $\rho(0)$.

Definition 2.1.23. Let S_g^b be a negative Euler characteristic surface. A *hexagon decomposition* of a surface S_g^b is a maximal collection of pairwise non-homotopic and disjoint simple essential arcs on S_g^b .

Remark 2.1.24. There always exists a hexagon decomposition on a negative Euler characteristic surface. The cardinality of a hexagon decomposition is $6g + 3b - 6$ and the hexagon decomposition cuts S_g^b into $4g + 2b - 4$ hexagons. See [Ush99].

When we endow S_g^b with a hyperbolic metric, and cut the surface along the orthogeodesic representatives of a hexagon decomposition, the surface decomposes into right-angled hexagons.

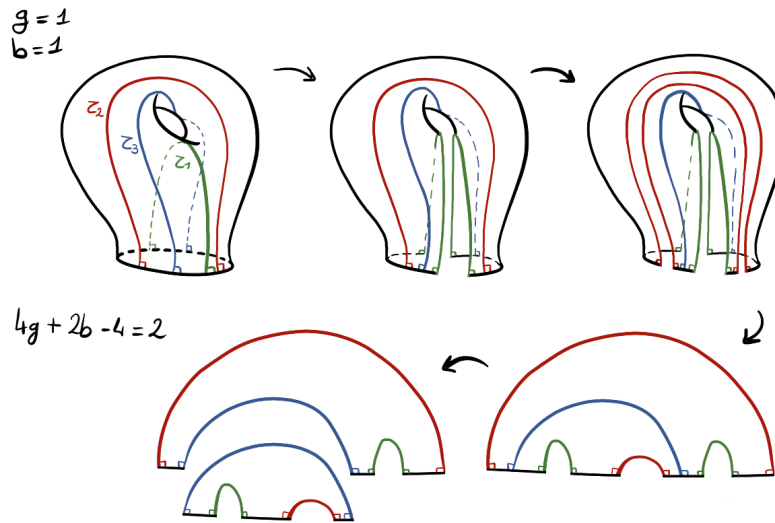


Figure 2.1: Example of a hexagon decomposition of the one-holed torus.

There is a link between pants and hexagon decompositions through the concept of double of surfaces.

Definition 2.1.25. Let S_g^b be a surface of genus g with $b > 0$ boundary components. The *double* of S_g^b is the surface obtained by gluing two copies of S_g^b along their corresponding boundary components. When S_g^b is endowed with a hyperbolic metric with geodesic boundary components, the double of S_g^b is endowed with the hyperbolic metric which coincides with the hyperbolic metric of S_g^b on each copy of S_g^b and such that, the reflection R between the two copies of S_g^b along their boundary is an isometry.

Let a be an arc of S_g^b . We call the *double of a* the closed curve defined as the union of the two copies of a on the double of S_g^b .

Remark 2.1.26. The double of S_g^b is the surface S_{2g+b-1} of genus $2g + b - 1$ with no boundary.

Lemma 2.1.27. Let $\mathcal{H}$ be a hexagon decomposition of S_g^b . Denote by S_{2g+b-1} the double of S_g^b and by R the reflection associated to it. Then, $\mathcal{H} \cup R(\mathcal{H})$ is a pants decomposition on S_{2g+b-1} .

Proof. The double of every simple arc of a hexagon decomposition forms a simple closed curve of S_{2g+b-1} . By construction, $\mathcal{H} \cup R(\mathcal{H})$ is a set of $6g + 3b - 6$ pairwise non-homotopic, disjoint, essential, simple, closed curves on S_{2g+b-1} . Therefore, it is a maximal collection of such curves, and hence a pants decomposition. $\square$

When talking about pairs of pants, we always assume that they have geodesic boundary and we will always consider geodesic pants decomposition and geodesic hexagon decomposition. Hyperbolic surfaces always admit pants decomposition but cutting a hyperbolic surface along a pants decomposition does not always result into a collection of pair of pants. It does however, when we look at its convex core.

Definition 2.1.28. Let X be a hyperbolic surface and Γ its Fuchsian group. The *convex core* of X , denoted $C(X)$, is the quotient of the convex hull of $\Lambda(\Gamma)$ by Γ .

Remark 2.1.29. A hyperbolic surface X is of the first kind if and only if it is equal to its convex core.

When instead we want to build a hyperbolic surface from a collection of pair of pants glued together, we use the following notion.

Definition 2.1.30. A hyperbolic surface X is said to be *geodesically complete* if the domain of every geodesic segment $\gamma : I \rightarrow X$ can be extended from the interval I to the whole $\mathbb{R}$.

Each pair of pants admits a natural geodesic completion to a geodesically complete hyperbolic surface, obtained by gluing a funnel to each boundary component. More generally, if X' is a hyperbolic surface constructed from finitely many pairs of pants glued along their boundary geodesics, then there is a unique geodesic completion X obtained by attaching funnels to the remaining boundary components. Geodesic completion is more subtle in the infinite-type case. For this, we rely on the following theorem [BS19, Theorem 3.4].

Theorem 2.1.31. Let X be a hyperbolic surface constructed from gluing pairs of pants that form a pants decomposition of X . Then the geodesic completion of X is attained by adding funnels and closed hyperbolic half-planes. Conversely, any geodesically complete hyperbolic surface is the geodesic completion of a hyperbolic surface X constructed from gluing pairs of pants that form a pants decomposition of X .

We can also define the geometric intersection number of closed curves and arcs.

Definition 2.1.32. Let α and β be closed curves or arcs on a surface S . The *geometric intersection number* $i(\alpha, \beta)$ between α and β is the minimum number of intersections counted with multiplicities between curves/arcs in their homotopy class (relative to the boundary).

Note that, if α and β are different closed geodesics or orthogeodesics $i(\alpha, \beta) = |\alpha \cap \beta|$.

We can define coordinates in the space of multi-curve that topologically identified each integral multi-curve.

Definition 2.1.33. Let $\mathcal{P} = \{\alpha_1, \dots, \alpha_{3g-3+b+n}\}$ be a pants decomposition on $S_g^{b,n}$. For all multi-curve γ we define its Dehn-Thurston coordinates as $(m_i, t_i)_{i=1}^{3g+b+n-3}$ with m_i the intersection number between γ and α_i , and t_i the twisting number of γ around α_i as defined in [LS04].

Let $Z(\mathcal{P})$ be the set of $(m_i, t_i)_{i=1}^{3g-3+b+n} \in (\mathbb{Z}_+ \times \mathbb{Z})^{3g-3+b+n}$ satisfying the following conditions:

- If $m_i = 0$, then $t_i \geq 0$,
- If $\alpha_{i_1}, \alpha_{i_2}$ and α_{i_3} are the boundary components of a pair of pants in $S_g^{b,n}$, then $m_{i_1} + m_{i_2} + m_{i_3}$ is even.

Then we have the Dehn theorem that guarantees that Dehn-Thurston coordinates uniquely determine integral multi-curves [PH92].

Theorem 2.1.34. For any pair of pants decomposition $\mathcal{P}$ on $S_g^{b,n}$, the map

$$DT : \mathcal{ML}_{g,b+n}(\mathbb{Z}) \rightarrow Z(\mathcal{P})$$

is a bijection.

2.2 Deformation space and coordinates

In this section we describe different spaces in which hyperbolic surfaces may live as well as ways to geometrically identify them. Let S be an orientable topological surface of negative Euler characteristic.

Definition 2.2.1. A *hyperbolic structure* is a pair (X, ϕ) with $\phi : S \rightarrow X$ a diffeomorphism and X a hyperbolic surface with geodesic boundary. The diffeomorphism ϕ is called the *marking*.

Two hyperbolic structures (X, ϕ) and (Y, ψ) are homotopic if there is an isometry $I : X \rightarrow Y$ such that the markings $I \circ \phi : S \rightarrow Y$ and $\psi : S \rightarrow Y$ are homotopic.

Definition 2.2.2. The *Deformation space* $\text{Def}(S)$ of S is the set of homotopy classes of hyperbolic structures on S .

We denote homotopy classes of hyperbolic surfaces in the Deformation space as χ or Υ . Let γ be a curve on S , by abuse of notation the curve $\phi(\gamma)$ on $\chi = [(X, \phi)]$ is called the curve γ on χ . We might still use the notation $\phi(\gamma)$ when it is important for the clarity.

Definition 2.2.3. The *mapping class group* $\mathcal{MCG}(S)$ of S is the quotient of the group of orientation-preserving homeomorphisms of S by the subgroup of homeomorphisms isotopic to the identity:

$$\mathcal{MCG}(S) = \text{Homeo}^+(S) / \text{Homeo}_0(S).$$

In the case of infinite-type surfaces we also use the term *big mapping class group*. We denote $\mathcal{PMCG}(S)$ and call pure mapping class group the finite index subgroup of $\mathcal{MCG}(S)$ consisting of exactly the elements which fix each boundary component and each puncture of S individually. See [FM12] for more on the mapping class groups and [AV20] for more on big mapping class groups.

Definition 2.2.4. The *moduli space* $\mathcal{M}(S)$ is the space of isometry classes of hyperbolic surfaces homeomorphic to S .

The moduli space is the quotient space

$$\mathcal{M}(S) = \text{Def}(S) / \mathcal{MCG}(S)$$

and the Deformation space is the universal cover of $\mathcal{M}(S)$.

The following functions on $\text{Def}(S)$ will be used throughout this paper. We start with the length function.

Definition 2.2.5. For any essential closed curve/arc α on S and any hyperbolic surface $\chi \in \text{Def}(S)$, the *length function* $\ell_\chi(\alpha)$ is the length of the shortest curve/arc in the homotopy class of α (relative to the boundary if α is an arc) on χ . In other words, it is the length of the geodesic/orthogeodesic representative of the homotopy class of α . If $\mathcal{C}$ is a collection of curves and arcs, we define $\ell_\chi(\mathcal{C}) = \sum_{\alpha \in \mathcal{C}} \ell_\chi(\alpha)$ and if $\gamma = \sum_i^k a_i \gamma_i$ is a multi-curve, we define $\ell_\chi(\gamma) = \sum_i^k a_i \ell_\chi(\gamma_i)$.

This function is well defined as there is only one geodesic/orthogeodesic representative in each homotopy class [Bus10, Theorem 1.5.2 and 1.6.6]. We may write $\ell_X(\alpha)$ with $X \in \mathcal{M}(S)$ instead of its lift $\chi \in \text{Def}(S)$ when there is no ambiguity on the marking of χ . When there is no ambiguity about the surface, we may just write $\ell(\alpha)$.

On finite surfaces $S_g^{b,n}$, we can use Dehn-Thurston coordinates to define another kind of length with respect to a pants decomposition: the combinatorial length on integral multi-curves.

Definition 2.2.6. Let $\mathcal{P} = \{\alpha_1, \dots, \alpha_{3g-3+b+n}\}$ be a pants decomposition on $S_g^{b,n}$ and γ an integral multi-curve on $S_g^{b,n}$ of Dehn-Thurston coordinates $(m_i, t_i)_{i=1}^{3g-3+b+n}$. Its combinatorial length on χ is

$$\mathcal{L}_{\mathcal{P}}(\chi, \gamma) = \sum_{i=1}^{3g-3+b+n} (m_i S(\ell_\chi(\alpha_i)) + |t_i| \ell_\chi(\alpha_i))$$

with $S(x) = \sinh^{-1} \left(\frac{1}{\sinh(x/2)} \right)$.

The length function also allow us to state the following result by Parlier [Par23] on pants decomposition of finite area surfaces.

Theorem 2.2.7. Let X be a finite area hyperbolic surface, possibly with geodesic boundary ∂X . Then X admits a pants decomposition where each curve is of length at most

$$B = \max\{\ell(\partial X), \text{area}(X)\}.$$

Another function is a function that depends on a pants decomposition fixed on S , namely the twist function. Let us fix a pants decomposition $\mathcal{P}$ on S . Let $\mathcal{P}'$ be a collection of disjoint simple closed curves on S such that for each pair of pants P in $S \setminus \mathcal{P}$, $\mathcal{P}' \cap P$ consist of 3 arcs each connecting a different pair of boundary components of P . Such a system of curves always exists regardless of the choice of $\mathcal{P}$ and we always fix one along fixing a pants decomposition.

Definition 2.2.8. For any fixed pants decomposition $\mathcal{P}$, any curve $\alpha \in \mathcal{P}$ and any hyperbolic structure $\chi = (X, \varphi) \in \text{Def}(S)$, the *twist function* $t_\chi(\alpha)$ is defined as follows.

Let $\alpha \in \mathcal{P}$, we give an orientation to α and denote P_l and P_r the pair of pants in $S \setminus \mathcal{P}$ respectively on the left and on the right of α (they may be the same pair of pants). Let η be one of the two components of $(P_l \cup P_r \cup \alpha) \cap \mathcal{P}'$ that intersect α .

Let δ be the geodesic representative of $\varphi(\alpha)$ on X . Let τ_r and τ_l be the orthogeodesics in $\varphi(P_r)$ and $\varphi(P_l)$ between δ and the boundary component hit by $\varphi(\eta)$. Denote by p_r and p_l the points on δ hit by τ_r and τ_l . Relatively to the boundary of $\varphi((P_r \cup P_l \cup \alpha))$, the arc $\varphi(\eta)$ is homotopic to $\tau_r \cdot \delta^k \cdot \tau_l$ for some $k \in \mathbb{Z}$. We define the *twist function* as

$$t_\chi(\alpha) = k + \frac{d(p_r, p_l)}{\ell_\chi(\alpha)}$$

with $d(p_r, p_l)$ the signed distance between p_r and p_l along the orientation of $\varphi(\alpha)$.

Now that the twist function is defined, we can relate the combinatorial length to the length function through the following theorem by Mirzakhani [Mir08, Proposition 3.5].

Theorem 2.2.9. Let $\mathcal{P}$ be a pants decomposition on $\chi \in \text{Def}(S_g^{b,n})$ such that for all $\alpha \in \mathcal{P}$, we have $\ell_\chi(\alpha) \leq L$ and $t_\chi(\alpha) \in [0, 1]$. Then for any integral multi-curve on χ , we have

$$\frac{1}{c(L)} \mathcal{L}_\mathcal{P}(\chi, \gamma) \leq \ell_\chi(\gamma) \leq c(L) \mathcal{L}_\mathcal{P}(\chi, \gamma), \quad (2.1)$$

with c that only depends on L, g, b and n .

The length function and the twist function provide different sets of coordinates that allow us to identify a hyperbolic surface in the Deformation space [ALP⁺11] [FM12].

Theorem 2.2.10. Let S be an orientable surface with negative Euler characteristic, b boundary components and n punctures. Given a pants decomposition $\mathcal{H} = \{\alpha_1, \dots, \alpha_N\}$ on S , the map

$$\varphi_\mathcal{P} : \text{Def}(S) \rightarrow \mathbb{R}_+^{N+b} \times \mathbb{R}^N$$

defined by

$$\varphi_\mathcal{P}(\chi) = (\ell_\chi(\alpha_N), \dots, \ell_\chi(\alpha_1), \dots, \ell_\chi(\alpha_b), t_\chi(\alpha_1), \dots, t_\chi(\alpha_N))$$

is a homeomorphism.

Note that if $S = S_g^{b,n}$ is a finite-type surface, then $N = 2g + b + n - 2$. Otherwise $N = \infty$ and we call the Deformation space infinite-dimensional Deformation space.

Definition 2.2.11. For a fixed pants decomposition $\mathcal{P} = \{\alpha_1, \dots, \alpha_N\}$, we call

$$\varphi_{\mathcal{P}}(\cdot) = (\ell_*(\alpha_N), \dots, \ell_*(\alpha_N), \ell_*(\alpha_1), \dots, \ell_*(\alpha_b), t_*(\alpha_1), \dots, t_*(\alpha_N))$$

Fenchel-Nielsen coordinates function.

We then use the Fenchel-Nielsen coordinates to define the topology on the Deformation space. Note that modulo 1 on any twist parameters, the Fenchel-Nielsen coordinates identify two isometric surfaces in the Deformation space.

For finite-type surfaces with geodesic boundary and without any puncture, an alternative set of coordinates can be defined using hexagon decomposition instead of pants decomposition. A right-angled hexagon is determined up to isometry by the lengths of three pairwise non-consecutive sides [Rat19, Theorem 3.5.14]. Thus, if we fix a hexagon decomposition on S_g^b , the lengths of the orthogeodesic representatives of the hexagon decomposition on S_g^b determine a hyperbolic surface in the Deformation space. Furthermore, Ushijima showed in [Ush99, Theorem 4.1] the following theorem:

Theorem 2.2.12. Given a hexagon decomposition $\mathcal{H} = \{\alpha_1, \dots, \alpha_{6g+3b-6}\}$ on S_g^b , the map

$$\varphi_{\mathcal{H}} : \text{Def}(S_g^b) \rightarrow \mathbb{R}_+^{6g+3b-6}$$

defined by

$$\varphi_{\mathcal{H}}(\chi) = (\ell_{\chi}(\alpha_1), \dots, \ell_{\chi}(\alpha_{6g+3b-6}))$$

is a homeomorphism.

Definition 2.2.13. For a fixed hexagon decomposition $\mathcal{H} = \{\alpha_1, \dots, \alpha_{6g+3b-6}\}$, we call

$$\varphi_{\mathcal{H}}(\cdot) = (\ell_*(\alpha_1), \dots, \ell_*(\alpha_{6g+3b-6}))$$

Ushijima coordinates function and denote by

$$\chi^{\omega} = \varphi_{\mathcal{H}}^{-1}(\omega)$$

the surface in $\text{Def}(S_g^b)$ associated to $\omega \in \mathbb{R}_+^{6g+3b-6}$.

For finite-type surfaces, the Deformation space of S_g^b admits a real analytic structure and the Fenchel-Nielsen coordinates are real analytic (independently from the pants decomposition) [Abi80]. For infinite-type surfaces, it is also possible to define an analytic structure on the Deformation space, however this structure may not be compatible anymore with the Fenchel-Nielsen coordinates, see [Hub06] for more information on this analytic structure.

By identifying $\text{Def}(S_g^b)$ with a real analytic subvariety of $\text{Def}(S_{2g+b-1})$, we obtain that this structure is compatible with the one given by Ushijima's coordinates (independently from the hexagon decomposition). Fix a hexagon decomposition $\mathcal{H} = \{\alpha_1, \dots, \alpha_{6g+3b-6}\}$ on S_g^b . By Lemma 2.1.27 this hexagon decomposition induces a pants decomposition on the double S_{2g+b-1} of S_g^b . We denote it $\mathcal{P} = \{\alpha'_1, \dots, \alpha'_{6g+3b-6}\}$ where α'_i is the double of α_i . We denote by $D(S_g^b)$ the subset of surfaces in $\text{Def}(S_{2g+b-1})$ that are doubles of surfaces in $\text{Def}(S_g^b)$, which is identified with $\text{Def}(S_g^b)$. We have

$$D(S_g^b) = \{\chi \in \text{Def}(S_{2g+b-1}) \mid t_{\alpha'_i}(\chi) = 0, 1 \leq i \leq 6g + 3b - 6\}$$

with $t_{\alpha'_i}(\cdot)$ the twists parameters in the Fenchel-Nielsen coordinates. The length parameters are real analytic on $D(S_g^b)$ and by identifying $\text{Def}(S_g^b)$ with $D(S_g^b)$, Ushijima's coordinates

$$(\ell(\alpha_1), \dots, \ell(\alpha_{6g+3b-6})) = \left(\frac{\ell(\alpha'_1)}{2}, \dots, \frac{\ell(\alpha'_{6g+3b-6})}{2} \right)$$

are real analytic. This does not depend on the hexagon decomposition. Through the same process with a pants decomposition on S_{2g+b-1} induced by a pants decomposition on S_g^b , we show that Fenchel-Nielsen coordinates are real analytic for the same analytic structure as Ushijima's coordinates.

Lemma 2.2.14. *For any arc β , the function*

$$\begin{aligned} \ell_*(\beta) : \text{Def}(S_g^b) &\rightarrow \mathbb{R}^+ \\ \chi &\mapsto \ell_\chi(\beta) \end{aligned}$$

is real analytic on $\text{Def}(S_g^b)$.

Proof. Let β be an arc on S_g^b and β' be the double of β on S_{2g+b-1} . By [Bus10, Lemma 10.2.3] (see also [MP08, Proposition 2.4]), we know that for any closed curve β' , the function $\ell_*(\beta')$ is real analytic on $\text{Def}(S_{2g+b-1})$ and thus on $D(S_g^b)$. By identifying $D(S_g^b)$ and $\text{Def}(S_g^b)$, we obtain that $\ell_*(\beta) = \frac{\ell_*(\beta')}{2}$ is real analytic on $\text{Def}(S_g^b)$. $\square$

Definition 2.2.15. Given two surfaces $\chi, \Upsilon \in \text{Def}(S_g^b)$ and $K \geq 1$, a homeomorphism $\phi : \chi \rightarrow \Upsilon$ is said to be K -quasiconformal if its distributional derivatives are locally in L_2 and

$$|\phi_{\bar{z}}| \leq \frac{K-1}{K+1} |\phi_z| \text{ a.e.}$$

In particular ϕ is said to be *conformal* if $K = 1$.

In the case of finite-type surfaces, we can use this notion of K -quasiconformality to define a metric on the Deformation space. Indeed, for any $\chi, \Upsilon \in \text{Def}(S_g^{b,n})$, there exists a unique K -quasiconformal map $\phi_{\chi, \Upsilon}$ between χ and Υ with K minimal (see [Abi80] for more details). The Deformation metric d_{Def} is defined by $d_{\text{Def}}(\chi, \Upsilon) = \frac{1}{2} \log(K)$; see [FM12].

This metric is compatible with the metric derived from the Fenchel-Nielsen and Ushijima coordinates. Let us state an extension of Wolpert's lemma [FM12, 12.3.2] to any closed curve, simple or not, due to Buser [Bus10, Theorem 6.4.3].

Lemma 2.2.16 (Wolpert's lemma). *Let $\phi : \chi_1 \rightarrow \chi_2$ be a K -quasiconformal homeomorphism between two hyperbolic surfaces χ_1 and χ_2 . For any isotopy class c of a closed curve in χ_1 , the following inequalities hold:*

$$\frac{\ell_{\chi_1}(c)}{K} \leq \ell_{\chi_2}(\phi(c)) \leq K \ell_{\chi_1}(c).$$

The previous lemma applied to the double of the surface implies the following.

Lemma 2.2.17 (Orthogeodesic Wolpert's lemma). *Let $\phi : \chi_1 \rightarrow \chi_2$ be a K -quasiconformal homeomorphism between two hyperbolic surfaces χ_1 and χ_2 . For any isotopy class c of arc in χ_1 , the following inequalities hold:*

$$\frac{\ell_{\chi_1}(c)}{K} \leq \ell_{\chi_2}(\phi(c)) \leq K \ell_{\chi_1}(c).$$

Then, by definition of the Deformation metric, for any $\chi_1, \chi_2 \in \text{Def}(S_g^b)$ with $d_{\text{Def}}(\chi_1, \chi_2) = \log(K)/2$, there is a K -quasiconformal homeomorphism between χ_1 and χ_2 . By Lemma 2.2.17, for any $\chi_1, \chi_2 \in \text{Def}(S_g^b)$ with $d_{\text{Def}}(\chi_1, \chi_2) \leq \log(K)/2$ and any isotopy class c of simple arcs on S_g^b , we have $\frac{\ell_{\chi_1}(c)}{K} \leq \ell_{\chi_2}(c) \leq K\ell_{\chi_1}(c)$.

In the case of infinite-type surfaces, the notion of metric is a bit more complicated. Different metrics that coincide in the finite-type case do not coincide anymore. Moreover, in the infinite-type case, the Deformation space may not be connected anymore. For more information on this topic see [LP11] and [Mat14].

2.3 Hyperbolic geometry properties

A lot of properties on hyperbolic surfaces come from the geometry of the hyperbolic plane. In this section we list some of these very useful properties that hold for any hyperbolic surface.

First, let us define the shear of two ideal triangles with a shared side.

Definition 2.3.1. Consider two hyperbolic ideal triangles Δ_1 and Δ_2 of disjoint interiors with one common edge g . We orient g such that Δ_1 is to its left. Consider the two orthorays based at g and going to the vertices of Δ_1 and Δ_2 opposite to g . The *shear* $s(g)$ of the configuration (Δ_1, Δ_2) is the signed distance (with respect to the orientation of g) between the feet of the orthorays in Δ_1 and Δ_2 .

It is straightforward to verify that the shear of the configuration (Δ_2, Δ_1) coincides with that of (Δ_1, Δ_2) .

Then, let us state several properties on geodesics and orthogeodesics using the length function. The following result is shown in [Bus10, Theorem 4.2.1].

Theorem 2.3.2. *Let X be a hyperbolic surface. Then every non-simple closed geodesic on X has length greater than 1.*

By doubling the surface, we deduce

Corollary 2.3.3. *Let X be a hyperbolic surface. Then every non-simple orthogeodesic on X has length greater than $\frac{1}{2}$.*

Proof. Let us take the doubled surface X' of X . For any non-simple orthogeodesic α on X , the doubled of α denoted by α' , is also a non-simple closed geodesic on X' . By Theorem 2.3.2, the length of α' is greater than 1. Since $\ell(\alpha') = 2\ell(\alpha)$, the length of α is greater than $\frac{1}{2}$. $\square$

We also have:

Lemma 2.3.4 (Half-collar lemma). *Let Y be a pair of pants with boundary geodesics $\gamma_1, \gamma_2, \gamma_3$. The sets*

$$\mathcal{C}^*[\gamma_i] = \{p \in Y \mid \sinh(\text{dist}(p, \gamma_i)) \sinh(\tfrac{1}{2}\ell(\gamma_i)) \leq 1\}$$

for $i = 1, 2, 3$ are pairwise disjoint and each of them is homeomorphic to a cylinder.

Lemma 2.3.5 (Collar lemma). *Let α and β be two distinct closed geodesics on a hyperbolic surface X such that $\alpha \cap \beta \neq \emptyset$. If β is simple, then*

$$\sinh\left(\frac{\ell(\alpha)}{2}\right) \sinh\left(\frac{\ell(\beta)}{2}\right) > 1.$$

The proofs of Lemmas 2.3.4 and 2.3.5 can be found in [Bus10]. We can state a version of the collar lemma with orthogeodesics instead of closed geodesics, which can be deduced by doubling the surface.

Lemma 2.3.6 (OrthoCollar lemma). *Let α and β be two distinct orthogeodesics on a hyperbolic surface X such that $\alpha \cap \beta \neq \emptyset$. If β is simple, then*

$$\sinh(\ell(\alpha)) \sinh(\ell(\beta)) > 1.$$

Proof. Let X' be the doubled surface of X . Denote by δ' and γ' the doubles of the orthogeodesics δ and γ . The geodesics δ' and γ' intersect each other and are distinct as it is the case for δ and γ . Moreover, if γ is simple, then γ' is also simple. By Lemma 2.3.5, we have

$$\sinh\left(\frac{\ell(\gamma')}{2}\right) \sinh\left(\frac{\ell(\delta')}{2}\right) > 1.$$

Since $\ell(\gamma') = 2\ell(\gamma)$ and $\ell(\delta') = 2\ell(\delta)$, we derive

$$\sinh(\ell(\gamma)) \sinh(\ell(\delta)) > 1.$$

□

Finally, let us state several hyperbolic trigonometry formulas, where the proofs can be found in [Bus10].

Lemma 2.3.7. *For any right-angled hexagon with consecutive sides $\beta, c, \alpha, b, \gamma, a$, we have*

$$\cosh(c) = \sinh(a) \sinh(b) \cosh(\gamma) - \cosh(a) \cosh(b). \quad (2.2)$$

For every trirectangle with sides labelled as in Figure 2.2, the following relation is true:

$$\cos(\varphi) = \tanh(\sigma) \tanh(\tau). \quad (2.3)$$

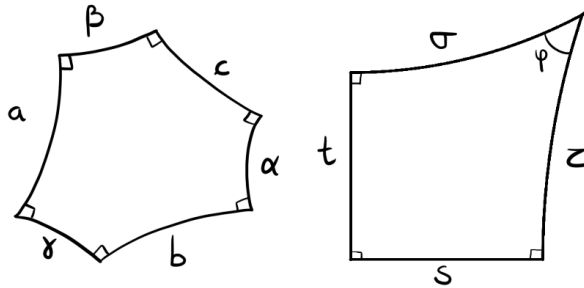


Figure 2.2: Right-angled hexagon and trirectangle

Definition 2.3.8. A Saccheri quadrilateral is a quadrilateral $ABCD$ with two right angles, at B and C , such that $[AB] = [CD]$. The side BC is called the base of the Saccheri quadrilateral.

As already mentioned, any right-angled hexagon is determined by the lengths of three pairwise disjoint sides. A right-angled octagon can be obtained by gluing two right-angled hexagons along one side. Thus, any right-angled octagon is determined by the length of four pairwise disjoint sides and the length of one orthogonal arc between opposite sides. In the following lemma, we will see how to compute the length of the orthogonal between the last two other sides.

Lemma 2.3.9 (Right-angled octagon). *We define the function $f_{\text{octa}} : \mathbb{R}_+^5 \rightarrow \mathbb{R}_+$ given by*

$$f_{\text{octa}}(a, x, y, z, t) = -\cosh(x) \cosh(t) + \sinh(x) \sinh(t) \times \\ \cosh \left[\cosh^{-1} \left(\frac{\cosh(y) + \cosh(x) \cosh(a)}{\sinh(x) \sinh(a)} \right) \right. \\ \left. + \cosh^{-1} \left(\frac{\cosh(z) + \cosh(t) \cosh(a)}{\sinh(t) \sinh(a)} \right) \right].$$

For any right-angled octagon with four disjoint sides $\delta_1, \delta_2, \delta_3, \delta_4$ and two orthogonal arcs α and β joining two opposite sides different from the δ_i as in Figure 2.3, we have:

$$\cosh(\beta) = f_{\text{octa}}(\alpha, \delta_1, \delta_2, \delta_3, \delta_4).$$

Proof. We can decompose the octagon into two right-angled hexagons with three disjoint sides $(\beta, \delta_4, \delta_1)$ and $(\beta, \delta_2, \delta_3)$. We can also decompose this octagon into two other right-angled hexagons with three disjoint sides $(\alpha, \delta_1, \delta_2)$ and $(\alpha, \delta_3, \delta_4)$. The arc α decomposes the side between δ_1 and δ_4 into two segments x and x' such that x is a side of the hexagon $(\alpha, \delta_1, \delta_2)$ and x' is a side of the hexagon $(\alpha, \delta_3, \delta_4)$ as in Figure 2.3.

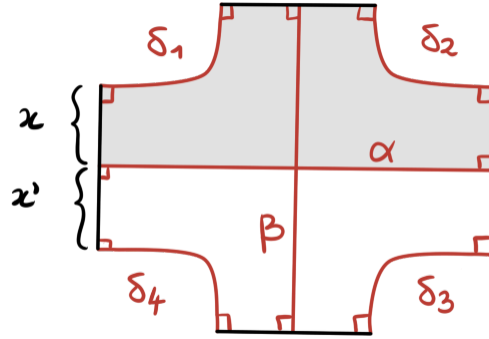


Figure 2.3: Right-angled octagon

We then apply (2.2) from Lemma 2.3.7 to the hexagons $(\alpha, \delta_1, \delta_2)$, $(\alpha, \delta_3, \delta_4)$ and $(\beta, \delta_4, \delta_1)$:

$$\begin{aligned} \cosh(x) &= \frac{\cosh(\delta_2) + \cosh(\delta_1) \cosh(\alpha)}{\sinh(\delta_1) \sinh(\alpha)} \\ \cosh(x') &= \frac{\cosh(\delta_3) + \cosh(\delta_4) \cosh(\alpha)}{\sinh(\delta_4) \sinh(\alpha)} \\ \cosh(\beta) &= \sinh(\delta_1) \sinh(\delta_4) \cosh(x + x') - \cosh(\delta_1) \cosh(\delta_4). \end{aligned}$$

We use the first two equalities to compute $x + x'$ and we use the expression in the last equality to conclude. $\square$

In [MM23, Lemma 3.2] and [MM23, Lemma 3.3], Masai and McShane prove the following results.

Lemma 2.3.10. *Let P be a pair of pants with boundary geodesics α, β, γ such that $\ell(\beta) \leq \ell(\alpha)$. Let τ be the unique simple orthogeodesic with both endpoints on γ as in Figure 2.4. Then, we have*

$$\sinh(\ell(\tau)/2) \leq \frac{\cosh(\ell(\alpha)/2)}{\sinh(\ell(\gamma)/4)}.$$

Let τ' be the unique orthogeodesic with one self-intersection, both endpoints on γ and that goes around α exactly once as in Figure 2.4. Then, we have

$$\ell(\tau') \leq \ell(\gamma) + 2\ell(\tau), \quad (2.4)$$

$$\text{and } \cosh\left(\frac{\ell(\tau')}{2}\right) = 2 \cosh\left(\frac{\ell(\alpha)}{2}\right) \cosh\left(\frac{\ell(\tau)}{2}\right). \quad (2.5)$$

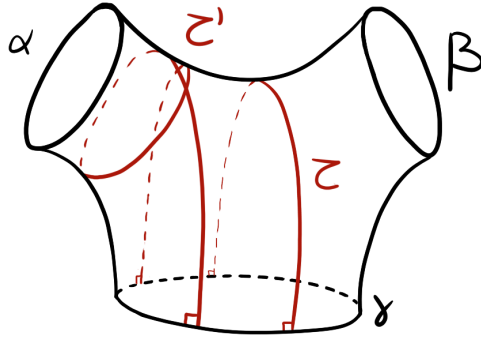


Figure 2.4: Orthogeodesics τ and τ' in P .

Notation

For two positive functions $a, b : \mathbb{N} \rightarrow \mathbb{R}$, we write

$$a \asymp b$$

if there exists a constant $K > 0$ such that for n sufficiently large

$$\frac{1}{K} \leq \frac{a(n)}{b(n)} \leq K.$$

Orthospectrum and simple orthospectrum rigidity

In this chapter, all hyperbolic surfaces are assumed to be compact with geodesic boundary and with at least one boundary component. Recall that we denote by S_g^b a genus g surface of negative Euler characteristic with b boundary components. Along this chapter we label the boundary components as $\{\gamma_0, \dots, \gamma_{b-1}\}$.

3.1 Orthospectrum, simple orthospectrum and subspaces of surfaces

Let us introduce the orthospectrum, an object analogous to the length spectrum and first defined by Basmajian in [Bas93].

Definition 3.1.1. The *orthospectrum* of a hyperbolic surface X is the multiset $\mathcal{O}(X)$ of lengths of orthogeodesics on X , counted with multiplicities.

Along with its definition, Basmajian showed the following in [Bas93].

Theorem 3.1.2. *The orthospectrum is discrete.*

We also define the simple orthospectrum.

Definition 3.1.3. The *simple orthospectrum* of a hyperbolic surface X is the multiset $\mathcal{O}_S(X)$ of lengths of simple orthogeodesics on X counted with multiplicities.

Among the properties of the orthospectrum, we highlight the following one by [Bas93], which shows that the boundary length of a hyperbolic surface is determined by its orthospectrum.

Theorem 3.1.4 (Basmajian's Identity.). *Let X be a compact hyperbolic surface with geodesic boundary. Then,*

$$\ell(\partial X) = \sum_{\ell \in \mathcal{O}(X)} F(\ell)$$

where $F(\ell) = 2 \sinh^{-1} \left(\frac{1}{\sinh(\ell)} \right)$.

Note that F is a positive decreasing function. For the simple orthospectrum, the theorem implies that

$$\ell(\partial X) > \sum_{\ell \in \mathcal{O}_S(X)} F(\ell).$$

Thus, X has at least one boundary component of length greater than $\frac{F(\text{osys}(X))}{b}$. Moreover, it is also possible to obtain an upper bound on $\ell(\partial X)$ thanks to a corollary of [Bav05, Théorème 1].

Corollary 3.1.5. *Let t be the orthosystole of X . Then*

$$\sinh(2t) \sinh \left(\frac{\ell(\partial X)}{24g + 12b - 24} \right) \leq \frac{1}{2}.$$

This gives us an upper bound

$$\ell(\partial X) \leq (24g + 12b - 24) \sinh^{-1} \left(\frac{1}{2 \sinh(2t)} \right).$$

Another interesting result related to the simple orthospectrum is the following theorem on counting simple arcs [Bel23] inspired by Mirzakhani's Theorem on counting simple geodesics [Mir08]: Two arcs are of the same type if they lie in the same $\mathcal{PMCG}(S_g^b)$ -orbit.

Theorem 3.1.6. *Let X be a hyperbolic structure with geodesic boundary on S_g^b . Let α_0 be a simple essential arc on S_g^b . Then there exist positive constants $c(\alpha_0)$ and $m(X)$ such that*

$$\lim_{L \rightarrow \infty} \frac{\#\{\alpha \text{ an arc of type } \alpha_0 \mid \ell_X(\alpha) \leq L\}}{L^{6g-6+2b}} = c(\alpha_0)m(X).$$

The constant $c(\alpha_0)$ only depends on the type of the arc α_0 and the topology of S_g^b . Moreover, there is a finite number of $\mathcal{PMCG}(S_g^b)$ -orbits thus a finite number of classes of simple arcs [ES22]. Thus we deduce the following corollary.

Corollary 3.1.7. *Let X be a hyperbolic structure with geodesic boundary on S_g^b . There exist positive constants $c(S_g^b)$ and $m(X)$ such that*

$$\lim_{L \rightarrow \infty} \frac{\#\{t \in \mathcal{O}_S(X) \mid t \leq L\}}{L^{6g-6+2b}} = c(S_g^b)m(X). \quad (3.1)$$

The constant $m(X)$ is the Thurston measure of the set of laminations of S_g^b of length smaller than 1. We do not need a precise definition of $m(X)$, for which we refer to [ES22]. Instead, we will use [Mir08, Proposition 3.1] adapted to surfaces with boundary, which can be deduced from [EPS20].

Theorem 3.1.8. *Let X be a hyperbolic structure with geodesic boundary on S_g^b . Then we have*

$$\lim_{L \rightarrow \infty} \frac{\#\{\gamma \in \mathcal{ML}_{g,b}(\mathbb{Z}) \mid \ell_X(\gamma) \leq L\}}{L^{6g-6+2b}} = m(X). \quad (3.2)$$

Thanks to this equality we can control the value of $m(X)$ with a lower bound on the systole of X and an upper bound on the length of the boundary of X . Define, for the rest of this chapter, $\varepsilon_m > 0$ to be small enough such that two non-homotopic simple closed curves shorter than ε_m are disjoint.

Proposition 3.1.9. *For all $g, b \geq 0$ and $B \geq 2\pi(2g+b-2)$, there exist two constants $c_1(B), c_2(B) > 1$ that depend only on B such that for all hyperbolic structure X with geodesic boundary on S_g^b and with $\ell(\partial X) \leq B$, we have*

$$\frac{1}{c_1(B)} \prod_{\gamma: \ell_X(\gamma) \leq \varepsilon_m} \frac{1}{\ell_X(\gamma)S(\ell_X(\gamma))} \leq m(X) \leq c_2(B) \prod_{\gamma: \ell_X(\gamma) \leq \varepsilon_m} \left(1 + \frac{1}{\ell_X(\gamma)S(\ell_X(\gamma))} + \frac{1}{\min\{\ell_X(\gamma), S(\ell_X(\gamma))\}}\right),$$

where the products are taken over all simple closed geodesics γ of length at most ε_m and $S(x) = \sinh^{-1}\left(\frac{1}{\sinh(x/2)}\right)$.

This proposition is an adaptation of [Mir08, Proposition 3.6] to hyperbolic surfaces with boundary, whose boundary length is bounded by B . We prove it for the sake of completeness using the following lemma by Mirzakhani [Mir08].

Lemma 3.1.10. *Given $x, y, L > 0$, consider the set $A_{x,y}(L)$ defined by*

$$A_{x,y}(L) = \{(m, n) \in \mathbb{Z}_+ \times \mathbb{Z}_+ \mid mx + ny \leq L\}.$$

Then for $L > 6 \max\{x, y\}$, we have

$$\frac{1}{12} \left(\frac{L^2}{xy}\right) \leq \#A_{x,y}(L). \quad (3.3)$$

And for $L > 1$, we have

$$\#A_{x,y}(L) \leq 3L^2 \left(1 + \frac{1}{xy} + \frac{1}{\min\{x, y\}}\right). \quad (3.4)$$

Proof of Theorem 3.1.9. Let $\alpha_1, \dots, \alpha_s$ be the curves on X of length less than ε_m . By Theorem 2.2.7, we can complete this family of curves into a pants decomposition $\mathcal{P} = \{\alpha_1, \dots, \alpha_s, \dots, \alpha_k\}$, where $k = 3g - 3 + b$, such that $\ell(\alpha_i) \leq B$. Since the Dehn twists on the curves in $\mathcal{P}$ are isometric transformations that change the twist parameters by 1 we can assume that the twist parameters are between 0 and 1. Thus, by inequality (2.1) we have

$$\begin{aligned} \#\{\gamma \in \mathcal{ML}_{g,b}(\mathbb{Z}) \mid c(B)\mathcal{L}_{\mathcal{P}}(X, \gamma) \leq L\} \\ \leq \#\{\gamma \in \mathcal{ML}_{g,b}(\mathbb{Z}) \mid \ell_X(\gamma) \leq L\} \\ \leq \#\{\gamma \in \mathcal{ML}_{g,b}(\mathbb{Z}) \mid \frac{1}{c(B)}\mathcal{L}_{\mathcal{P}}(X, \gamma) \leq L\}. \end{aligned}$$

Let us start by computing a lower bound for $m(X)$.

By definition of the combinatorial length and Theorem 2.1.34, we obtain

$$\#\{\gamma \in \mathcal{ML}_{g,b}(\mathbb{Z}) \mid c(B)\mathcal{L}_{\mathcal{P}}(X, \gamma) \leq L\} = \#\left\{(m_i, t_i)_{i=1}^k \in \mathcal{Z}(\mathcal{P}) \mid c(B) \left(\sum_{i=1}^k m_i S(\ell_X(\alpha_i)) + |t_i| \ell_X(\alpha_i) \right) \leq L\right\}.$$

Note that we have $\{(m_i, t_i)_{i=1}^k \mid t_i > 0, m_i \in 2\mathbb{Z}\} \subset \mathcal{Z}(\mathcal{P})$, thus

$$\begin{aligned} \#\left\{(m_i, t_i)_{i=1}^k \in (\mathbb{Z}_+ \times \mathbb{Z}_+)^k \mid c(B) \left(\sum_{i=1}^k 2m_i S(\ell_X(\alpha_i)) + t_i \ell_X(\alpha_i) \right) \leq L\right\} &\leq \#\{\gamma \in \mathcal{ML}_{g,b}(\mathbb{Z}) \mid \ell_X(\gamma) \leq L\} \\ \#\left\{(m_i, t_i)_{i=1}^k \in (\mathbb{Z}_+ \times \mathbb{Z}_+)^k \mid c(B) (2m_i S(\ell_X(\alpha_i)) + t_i \ell_X(\alpha_i)) \leq \frac{L}{k}\right\} &\leq \\ \prod_{i=1}^k \#\left\{(m_i, t_i) \in \mathbb{Z}_+ \times \mathbb{Z}_+ \mid 2m_i S(\ell_X(\alpha_i)) + t_i \ell_X(\alpha_i) \leq \frac{L}{c(B)k}\right\} &\leq \end{aligned}$$

By Inequality (3.3), for L large enough, we have

$$\prod_{i=1}^k \frac{1}{12} \left(\frac{L}{kc(B)} \right)^2 \frac{1}{2S(\ell_X(\alpha_i))\ell_X(\alpha_i)} \leq \#\{\gamma \in \mathcal{ML}_{g,b}(\mathbb{Z}) \mid \ell_X(\gamma) \leq L\}$$

We separate the product over the curves α_i into a product over the ones of length at most ε_m and a product over the other ones.

$$\begin{aligned} \frac{L^{6g-6+2b} \prod_{i=s+1}^k \frac{1}{S(\varepsilon_m)B}}{(24k^2 c(B)^2)^{3g-3+b}} \prod_{\gamma: \ell_X(\gamma) \leq \varepsilon_m} \frac{1}{\ell_X(\gamma) S(\ell_X(\gamma))} &\leq \#\{\gamma \in \mathcal{ML}_{g,b}(\mathbb{Z}) \mid \ell_X(\gamma) \leq L\} \\ \frac{L^{6g-6+2b}}{(24k^2 c(B)^2 S(\varepsilon_m) B)^{3g-3+b}} \prod_{\gamma: \ell_X(\gamma) > \varepsilon_m} \frac{1}{\ell_X(\gamma) S(\ell_X(\gamma))} &\leq \end{aligned}$$

We obtain

$$\frac{1}{c_1(B)} \prod_{\gamma: \ell_X(\gamma) \leq \varepsilon_m} \frac{1}{\ell_X(\gamma) S(\ell_X(\gamma))} \leq \frac{\#\{\gamma \in \mathcal{ML}_{g,b}(\mathbb{Z}) \mid \ell_X(\gamma) \leq L\}}{L^{6g-6+2b}}$$

and letting L go to infinity, we get

$$\frac{1}{c_1(B)} \prod_{\gamma: \ell_X(\gamma) \leq \varepsilon_m} \frac{1}{\ell_X(\gamma) S(\ell_X(\gamma))} \leq m(X).$$

Now, let us compute the upper bound on $m(X)$, recall that we have

$$\#\{\gamma \in \mathcal{ML}_{g,b}(\mathbb{Z}) \mid \ell_X(\gamma) \leq L\} \leq \#\{\gamma \in \mathcal{ML}_{g,b}(\mathbb{Z}) \mid \frac{1}{c(B)} \mathcal{L}_{\mathcal{P}}(X, \gamma) \leq L\}.$$

By definition of the combinatorial length and Theorem 2.1.34, we obtain

$$\begin{aligned} \#\{\gamma \in \mathcal{ML}_{g,b}(\mathbb{Z}) \mid \ell_X(\gamma) \leq L\} &\leq \#\left\{ (m_i, t_i)_{i=1}^k \in \mathcal{Z}(\mathcal{P}) \mid \frac{1}{c(B)} \left(\sum_{i=1}^k m_i S(\ell_X(\alpha_i)) + |t_i| \ell_X(\alpha_i) \right) \leq L \right\} \\ &\leq 2^k \#\left\{ (m_i, t_i)_{i=1}^k \in (\mathbb{Z}_+ \times \mathbb{Z}_+)^k \mid \frac{1}{c(B)} \left(\sum_{i=1}^k m_i S(\ell_X(\alpha_i)) + t_i \ell_X(\alpha_i) \right) \leq L \right\} \\ &\leq 2^k \#\left\{ (m_i, t_i)_{i=1}^k \in (\mathbb{Z}_+ \times \mathbb{Z}_+)^k \mid \frac{1}{c(B)} (m_i S(\ell_X(\alpha_i)) + t_i \ell_X(\alpha_i)) \leq L \right\} \\ &\leq \prod_{i=1}^k 2 \#\{(m_i, t_i) \in \mathbb{Z}_+ \times \mathbb{Z}_+ \mid m_i S(\ell_X(\alpha_i)) + t_i \ell_X(\alpha_i) \leq c(B)L\}. \end{aligned}$$

By Inequality (3.4), for L large enough, we have

$$\begin{aligned} \#\{\gamma \in \mathcal{ML}_{g,b}(\mathbb{Z}) \mid \ell_X(\gamma) \leq L\} &\leq \prod_{i=1}^k 6(c(B)L)^2 \left(1 + \frac{1}{S(\ell_X(\alpha_i)) \ell_X(\alpha_i)} + \frac{1}{\min\{S(\ell_X(\alpha_i)), \ell_X(\alpha_i)\}} \right) \\ &\leq L^{6g-6+2b} \prod_{i=1}^k 6c(B)^2 \left(1 + \frac{1}{S(\ell_X(\alpha_i)) \ell_X(\alpha_i)} + \frac{1}{\min\{S(\ell_X(\alpha_i)), \ell_X(\alpha_i)\}} \right). \end{aligned}$$

Hence,

$$\frac{\#\{\gamma \in \mathcal{ML}_{g,b}(\mathbb{Z}) \mid \ell_X(\gamma) \leq L\}}{L^{6g-6+2b}} \leq c_2(B) \prod_{\gamma: \ell_X(\gamma) \leq \varepsilon_m} \left(1 + \frac{1}{S(\ell_X(\alpha_i)) \ell_X(\alpha_i)} + \frac{1}{\min\{S(\ell_X(\alpha_i)), \ell_X(\alpha_i)\}} \right).$$

Letting L go to infinity, we obtain

$$m(X) \leq c_2(B) \prod_{\gamma: \ell_X(\gamma) \leq \varepsilon_m} \left(1 + \frac{1}{S(\ell_X(\alpha_i)) \ell_X(\alpha_i)} + \frac{1}{\min\{S(\ell_X(\alpha_i)), \ell_X(\alpha_i)\}} \right).$$

□

Going back and forth between the moduli space and the Deformation space seen as its cover is useful to get the best of both spaces. Let us define some interesting subsets and properties of the moduli space.

Definition 3.1.11. For $A, \varepsilon > 0$, set

$$\mathcal{M}_{A,\varepsilon}(S_g^b) = \{X \in \mathcal{M}(S_g^b) \mid \text{sys}(X) \geq \varepsilon \text{ and } A \geq \ell_X(\gamma_0), \dots, \ell_X(\gamma_{b-1}) \geq \varepsilon\}$$

Theorem 2.2.7 will help us derive a property of $\mathcal{M}_{A,\varepsilon}(S_g^b)$.

Theorem 3.1.12. For all $A \geq \varepsilon > 0$, $\mathcal{M}_{A,\varepsilon}(S_g^b)$ is compact.

This result was proven by Mumford [Mum71] in the closed case. In [LPST09], a different version of the theorem is stated. For the sake of completeness we give the proof for surfaces with boundary by adapting Farb and Margalit's proof in [FM12, Chap. 12]. With Corollary 3.1.5, we obtain an equivalence between Theorem 3.1.12 and [LPST09, Theorem 4.1].

Proof. Since $\mathcal{M}(S_g^b)$ inherits the Deformation metric from $\text{Def}(S_g^b)$, we just need to show that $\mathcal{M}_{A,\varepsilon}(S_g^b)$ is sequentially compact. Let (X_i) be a sequence in $\mathcal{M}_{A,\varepsilon}(S_g^b)$ and $\chi_i \in \text{Def}(S_g^b)$ a lift of X_i for all i .

To show that a subsequence of (X_i) converges in $\mathcal{M}_\varepsilon(S_g^b)$, we prove that for a fixed choice of Fenchel-Nielsen coordinates, we can choose lifts Υ_i of X_i to $\text{Def}(S_g^b)$ inside a rectangular compact set of the Euclidean space $\mathbb{R}_{>0}^{3g-3+2b} \times \mathbb{R}^{3g-3+b}$.

By Theorem 2.2.7, there is a pants decomposition $\mathcal{P}_i$ of S_g^b such that $\ell_{\chi_i}(\gamma) \in [\varepsilon, B]$ for all $\gamma \in \mathcal{P}_i$, where $B = \max\{bA, 2\pi(2g + b - 1)\}$.

Since there are a finitely many topological types of pants decompositions of S_g^b , we can choose a sequence (f_i) in $\mathcal{MCG}(S_g^b)$ such that, up to passing to a subsequence, $f_i(\mathcal{P}_i) = \mathcal{P}_1$. The hyperbolic structure $\Upsilon_i = f_i \cdot \chi_i$ is also a lift of X_i , whose length parameters in the Fenchel-Nielsen coordinates with respect to $\mathcal{P}_1$ are between ε and B .

Since the Dehn twists on the curves in $\mathcal{P}_1$ change the twist parameters by 1, there is a product h_i of Dehn twists on the curves of $\mathcal{P}_1$ such that the twist parameters of $h_i \cdot \Upsilon_i$ are between 0 and 1. Thus, the lifts $h_i \cdot \Upsilon_i$ of X_i are all inside a compact set. Therefore, there exists a converging subsequence which projects to a converging subsequence of (X_i) . $\square$

Then we state a consequence of Theorem 3.1.12.

Corollary 3.1.13. Let $A \geq \varepsilon > 0$. There exists a compact subset $\mathcal{Q}(A, \varepsilon) \subset \text{Def}(S_g^b)$ such that for each surface $\chi \in \text{Def}(S_g^b)$ with $\text{sys}(\chi) \geq \varepsilon$ and boundary component lengths between ε and A , there exists an isometric surface $\chi' \in \mathcal{Q}(A, \varepsilon)$.

Proof. By Theorem 3.1.12, the set $\mathcal{M}_{A,\varepsilon}(S_g^b)$ of hyperbolic surfaces X , up to isometry, with $\text{sys}(X) \geq \varepsilon$ and boundary component lengths between ε and A is compact. We set $\mathcal{Q}(A, \varepsilon)$ a compact lift of $\mathcal{M}_{A,\varepsilon}(S_g^b)$ to the Deformation space. By definition, any surface $\chi' \in \text{Def}(S_g^b)$ with $\text{sys}(\chi') \geq \varepsilon$ and boundary component lengths between ε and A is mapped by the cover to a surface $X' \in \mathcal{M}_{A,\varepsilon}(S_g^b)$. By construction, there is a lift χ of X' in $\mathcal{Q}(A, \varepsilon)$ and χ is isometric to χ' . $\square$

Finally, we state a corollary of Lemma 2.2.17.

Corollary 3.1.14. *Let $\mathcal{Q} \subset \text{Def}(S_g^b)$ be a compact subset. There exists a constant $q \geq 1$, which depends only on $\mathcal{Q}$, such that*

$$\frac{\ell_\chi(\beta)}{q} \leq \ell_{\chi'}(\beta) \leq q\ell_\chi(\beta)$$

for any $\chi, \chi' \in \mathcal{Q}$ and any arc β .

Proof. Let $q > 0$ such that $\text{diam } \mathcal{Q} = \frac{1}{2} \log(q)$. Then for any $\chi, \chi' \in \mathcal{Q}$, we have $d_{\text{Def}}(\chi, \chi') \leq \log(q)/2$. By Lemma 2.2.17, we have

$$\frac{\ell_\chi(\beta)}{q} \leq \ell_{\chi'}(\beta) \leq q\ell_\chi(\beta).$$

□

Using the analytic nature of the Deformation space, we also define some negligible subsets of $\text{Def}(S_g^b)$.

Definition 3.1.15. Define the relation of equivalence $\sim$ as follows: Let τ and τ' be two arcs on S_g^b . We set $\tau \sim \tau'$ if for all $\chi \in \text{Def}(S_g^b)$, we have $\ell_\chi(\tau) = \ell_\chi(\tau')$. Denote $[[\tau]]$ the class of equivalence of τ for the relation $\sim$.

Example 3.1.16. By symmetry of the pair of pants, the orthogeodesics τ and τ' in Figure 3.1 are in the same class $[[\tau]]$.

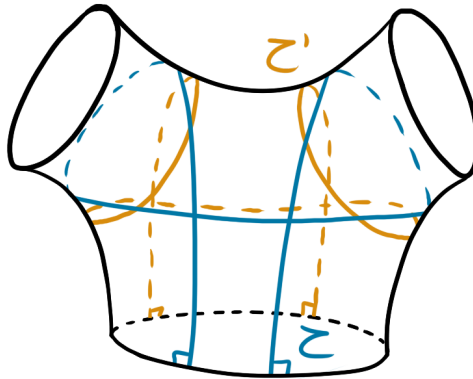


Figure 3.1: Example of two orthogeodesics in the same class of equivalence of $\sim$.

Curves with the same property also exist and have been studied, see Chris Leininger's paper [Lei03] and references there, in part [Hor72] and [Ran80].

We define the length of $[[\tau]]$ as $\ell_\chi([[\tau]]) = \ell_\chi(\tau)$. By definition of $\sim$, this length does not depend on the choice of representative τ .

Definition 3.1.17. Let $[[\tau]]$ and $[[\tau']]$ be two different equivalence classes of arcs. We define the set

$$\mathcal{E}([[\tau]], [[\tau']]) = \{\chi \in \text{Def}(S_g^b) \mid \ell_\chi([[\tau]]) = \ell_\chi([[\tau']])\}.$$

Proposition 3.1.18. *The set $\mathcal{E}([\tau], [\tau'])$ is a proper real analytic variety of $\text{Def}(S_g^b)$.*

Proof. By Lemma 2.2.14, the length function for any arc β , and thus for any class of arc $[[\beta]]$, is real analytic on $\text{Def}(S_g^b)$. Therefore, $\mathcal{E}([\tau], [\tau'])$ is a real analytic variety. Moreover, by definition of $\sim$, for $[[\tau]]$ and $[[\tau']]$ different, we have $\mathcal{E}([\tau], [\tau']) \neq \text{Def}(S_g^b)$. Thus $\mathcal{E}([\tau], [\tau'])$ is a proper real analytic variety of $\text{Def}(S_g^b)$. $\square$

In particular, this means that $\mathcal{E}([\tau], [\tau'])$ is negligible in $\text{Def}(S_g^b)$. As

$$\bigcup_{[[\tau]] \neq [[\tau']]} \mathcal{E}([\tau], [\tau'])$$

is a countable union, it is also a negligible subset of $\text{Def}(S_g^b)$.

3.2 A finite number of surfaces with the same orthospectrum or simple orthospectrum

Let us define, for any hyperbolic structure X on S_g^b , the sets

$$I(X) = \{Y \in \mathcal{M}(S_g^b) \text{ such that } \mathcal{O}(Y) = \mathcal{O}(X)\}.$$

and

$$I_S(X) = \{Y \in \mathcal{M}(S_g^b) \text{ such that } \mathcal{O}_S(Y) = \mathcal{O}_S(X)\}.$$

The goal of this section is to show that each one of these sets are finite.

In 2023, Masai and McShane showed the following theorem [MM23, Theorem 5.1]:

Theorem 3.2.1. *Let X be a hyperbolic structure on S_g^1 . Then the set $I(X)$ is finite.*

Their result is restricted to surfaces with one boundary component but it can be extended to surfaces with any number of boundary components b .

Theorem 3.2.2. *Let X be a hyperbolic structure on S_g^b with geodesic boundary. Then, the set $I(X)$ is finite.*

We will only detail parts of the proof that diverge from the proof of Masai and McShane. In particular, the proof can be separated into two steps: First, show that $I(X)$ is included in a compact of the form $\mathcal{M}_{A,\varepsilon}(S_g^b)$. Then show that $I(X)$ is itself compact and discrete thus finite. The difference between the proof of Theorem 3.2.1 and Theorem 3.2.2 lies in the first step, thus we will only show the first step of the proof. However, note that the second step of the proof in the case of the simple orthospectrum also works here.

Proof. To prove that $I(X)$ is included in a compact, we compute two constants $A > 0$ and $\varepsilon > 0$ that depend on the orthospectrum of X such that $I(X) \subset \mathcal{M}_{A,\varepsilon}(S_g^b)$ (which is compact by Theorem 3.1.12).

If X has a single boundary component γ_0 then, by Theorem 3.1.4, its length is determined by the orthospectrum of X and we can choose $A = \ell(\gamma_0) = \sum_{\ell \in \mathcal{O}(X)} F(\ell)$. When X has b boundary

components it is still possible to find an upper bound on their length by Corollary 3.1.5 and we can choose $A = (24g + 12b - 24) \sinh^{-1} \left(\frac{1}{2 \sinh(2 \text{osys}(X))} \right)$.

Then Masai and McShane argue by induction to compute a lower bound on the systole of any surface $Y \in \mathcal{I}(X)$. Their argument use an upper bound on the boundary length, the discreteness of the orthospectrum and a lower bound on the length of at least one of the boundary component. In the proof of Masai and McShane, the length of γ_0 is determined by the orthospectrum. In the case of a surface with b boundary components, the length of each of the boundary components is bounded by A . By Theorem 3.1.4, at least one of the boundary components has a length greater than $\varepsilon_0 = \frac{\sum_{\ell \in \mathcal{O}(X)} F(\ell)}{b}$. Then, we follow the same steps as Masai and McShane to compute a lower bound ε on the systole as well as a lower bound on the length of each boundary component. This is possible because each induction step does not depend on whether if the boundary components of the pair of pants correspond to non-peripheral curves on Y or to boundary components of Y . $\square$

In the case of the simple orthospectrum we can prove a result analogous to Theorem 3.2.1:

Theorem 3.2.3. *Let X be a hyperbolic structure on S_g^1 . Then the set $\mathcal{I}_S(X)$ is finite.*

Similarly, the proof can be done in the same two steps. However the way to prove each of these steps uses completely different techniques. Indeed, Masai and McShane's proof uses of some orthogeodesics with one self-intersection, which we cannot do in the case of the simple orthospectrum.

Proof. First, let us show that $\mathcal{I}_S(X)$ is included in $\mathcal{M}_{A,\varepsilon}(S_g^1)$ by computing for all $Y \in \mathcal{I}_S(X)$ an upper bound and a lower bound on the length of each boundary component and a lower bound on its systole.

By Corollary 3.1.5, we have a common upper bound A on the length on the boundary components of any $Y \in \mathcal{I}_S(X)$.

By Corollary 3.1.7, if $Y \in \mathcal{I}_S(X)$, we have

$$m(Y)c(S_g^1) = \lim_{L \rightarrow \infty} \frac{\#\{t \in \mathcal{O}_S(Y) \mid t \leq L\}}{L^{6g-4}} = \lim_{L \rightarrow \infty} \frac{\#\{t \in \mathcal{O}_S(X) \mid t \leq L\}}{L^{6g-4}} = m(X)c(S_g^1).$$

By Proposition 3.1.9, we have that $m(Y)$ goes to infinity as $\text{sys}(Y)$ goes to 0. As $m(Y)$ is constant, $\text{sys}(Y)$ cannot go to 0, i.e. there exists a constant $\varepsilon_1 > 0$ such that for all $Y \in \mathcal{I}_S(X)$, we have $\text{sys}(Y) \geq \varepsilon_1$.

Finally, as the surface only has one boundary component γ_0 , it follows from Basmajian's Identity that

$$\ell_Y(\gamma_0) \geq \varepsilon_2 = \sum_{\ell \in \mathcal{O}_S(X)} F(\ell) > 0$$

for all $Y \in \mathcal{I}_S(X)$. Denote $\varepsilon = \min(\varepsilon_1, \varepsilon_2)$. Then $\mathcal{I}_S(X) \subset \mathcal{M}_{A,\varepsilon}(S_g^1)$, which is compact.

For the second step of the proof, let us fix a hexagon decomposition $\mathcal{H}$ on S_g^1 and set $\varphi_{\mathcal{H}}$ as in Theorem 2.2.12.

Because $\mathcal{I}_S(X)$ is included in a compact subset of $\mathcal{M}(S_g^1)$, there is a lift $\tilde{\mathcal{I}}_S(X)$ of $\mathcal{I}_S(X)$ in $\text{Def}(S_g^1)$ which is also included in a compact C . We will show that $\tilde{\mathcal{I}}_S(X)$ is finite, and so is $\mathcal{I}_S(X)$.

Since $\tilde{I}_S(X)$ is included in a compact, for any sequence $\chi_n \in \tilde{I}_S(X)$, there is a subsequence such that $\chi_n \rightarrow \chi \in \mathbb{C}$. The map $\varphi_{\mathcal{H}}$ is continuous, so $\varphi_{\mathcal{H}}(\chi_n) \rightarrow \varphi_{\mathcal{H}}(\chi)$. We recall that for every n , we have $\mathcal{O}_S(\chi_n) = \mathcal{O}_S(X)$ and that by Theorem 3.1.2 the orthospectrum is discrete.

As, $\varphi_{\mathcal{H}}(\chi_n) \subset \mathcal{O}_S(X)$, we deduce from Wolpert's lemma 2.2.17 that there exists $N \in \mathbb{N}^*$ such that for all $n', m' > N$, $\varphi_{\mathcal{H}}(\chi_{n'}) = \varphi_{\mathcal{H}}(\chi_{m'})$. Thus, for all $n', m' > N$, $\chi_{n'} = \chi_{m'} = \chi$ and $\chi \in \tilde{I}_S(X)$. So $\tilde{I}_S(X)$ is compact and discrete, thus finite, and so is $I_S(X)$. □

3.3 Generic surfaces are determined by their orthospectrum and by their simple orthospectrum

In this section, we are going further in our characterization of the rigidity of the orthospectrum and the simple orthospectrum. We prove the following theorem:

Theorem 3.3.1. *Let $\mathcal{V}_g^b$ be the subset of all $\chi \in \text{Def}(S_g^b)$ for which there exists $\Upsilon \in \text{Def}(S_g^b)$ non-isometric to χ such that $\mathcal{O}(\chi) = \mathcal{O}(\Upsilon)$. Similarly, let $\mathcal{W}_g^1$ be the subset of all $\chi \in \text{Def}(S_g^1)$ for which there exists $\Upsilon \in \text{Def}(S_g^1)$ non-isometric to χ such that $\mathcal{O}_S(\chi) = \mathcal{O}_S(\Upsilon)$.*

Then, $\mathcal{V}_g^b$ is a negligible sets of $\text{Def}(S_g^b)$ and $\mathcal{W}_g^1$ is a negligible sets of $\text{Def}(S_g^1)$. Moreover, $\mathcal{W}_g^1$ is a proper local real analytic subvarieties of $\text{Def}(S_g^1)$.

This result is a version of Wolpert's Theorem [Wol79] for the orthospectrum and the simple orthospectrum instead of the length spectrum. We adapt the proof given by Buser in [Bus10, Chap. 10] to our case. Doing so, we do not require non-simple arcs, thus most of the proof is the same both for the orthospectrum and the simple orthospectrum. However there are still some key differences that will be highlighted.

We first set up all the context and notation, then we prove a lemma on the first lengths of the orthospectrum and simple orthospectrum, and finally we prove Theorem 3.3.1.

3.3.1 Set up and prerequisites

First let us state two propositions on two different kinds of open subsets of the Deformation space that will play a key role in the proof of Theorem 3.3.1. Each kind corresponds to the orthospectrum case and the simple orthospectrum case. These subsets will play the same role in the proof of Theorem 3.3.1 but the subtle differences between the two will influence the end of the proof.

In the case of the orthospectrum, there exist open balls in $\text{Def}(S_g^b)$ with the following property.

Proposition 3.3.2. *For all $\chi \in \text{Def}(S_g^b) \setminus \bigcup_{[[\tau]] \neq [[\tau']] \mathcal{E}([[\tau]], [[\tau']])$, there exists a constant $q > 1$ such that the open ball $\mathcal{U}_\chi$ of center χ and radius $\frac{\log(q)}{2}$ satisfies the following property: there exist constants $\varepsilon_\chi > 0$ and $A_\chi > 0$ such that for all $\Upsilon \in \mathcal{U}_\chi$ and $\Upsilon' \in \text{Def}(S_g^b)$ with $\mathcal{O}(\Upsilon') = \mathcal{O}(\Upsilon)$, there is a surface isometric to Υ' in $\mathcal{Q}(A_\chi, \varepsilon_\chi)$.*

Proof. First, let us define the following:

$$\delta(L) = \min_{t, t' \in \mathcal{O}(\chi)} \{t' - t \mid t < t' \leq L\}.$$

This quantity measures the smallest gap between lengths of orthogeodesics of length bounded by L . Since there are finitely many orthogeodesics of length shorter than any L , $\delta(L)$ is well defined for any $L > 0$ greater than the second shortest length in $\mathcal{O}(\chi)$.

To deploy this proof we first need to recall the details of the induction parts of the proof of Theorem 3.2.2, especially, how to compute a lower bound on the systole and the length of the boundary components of a surface χ' with its orthospectrum. Let $\chi' \in \text{Def}(S_g^b)$ such that $\mathcal{O}(\chi') = \mathcal{O}(\chi)$.

By Corollary 3.1.5, if $A = (24g + 12b - 24) \sinh^{-1} \left(\frac{1}{2 \sinh(2 \text{osys}(\chi))} \right)$, then $A \geq \ell(\partial\chi')$. By Theorem 3.1.4, at least one of the boundary components of χ' has a length greater than $\varepsilon_0 = \frac{\sum_{\ell \in \mathcal{O}(\chi)} F(\ell)}{b}$. We choose this boundary component to be γ_0 .

Define $\mathcal{P}$ a pants decomposition on χ' as in Theorem 2.2.7. The curves in $\mathcal{P}$ and the boundary components of χ' have length at most $B = \max\{A, \text{area}(\chi)\}$.

Construct a rooted graph G dual to the pants decomposition, as follows. Each vertex corresponds to a pair of pants in χ' given by the pants decomposition $\mathcal{P}$. The root corresponds to the pair of pants, denoted by P_0 , with γ_0 as one of its boundary components. A pair of vertices is joined by an edge each time the corresponding pairs of pants have a boundary component in common.

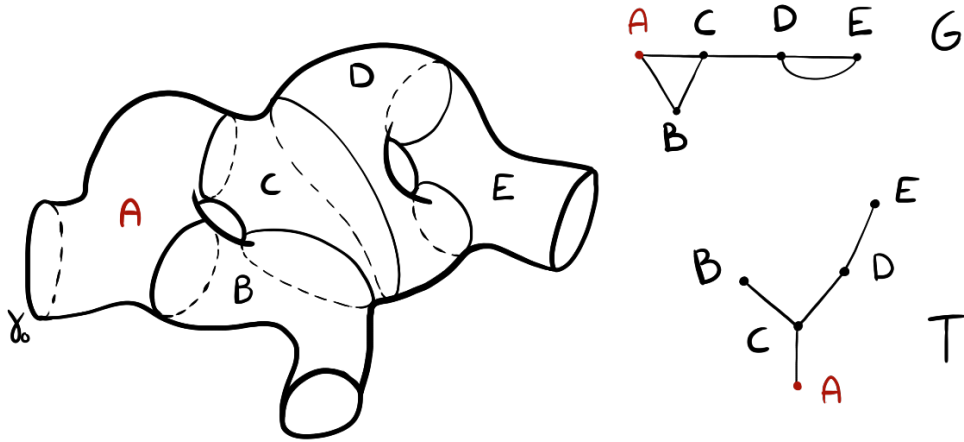


Figure 3.2: Example of a rooted graph G .

Choose a spanning rooted tree T in G . We show by induction on the distance to the root that we have a lower bound on the length of each boundary component of each pair of pants corresponding to the vertices.

The base case starts with P_0 . Label α_1 and β_1 the other two boundary components of P_0 . With Lemma 2.3.10 we compute an upper bound L_0 on two specific orthogeodesics τ_0 and τ'_0 (see Figure 3.3) that depend on ε_0 and B .

$$\ell(\tau_0) \leq 2 \sinh^{-1} \left(\frac{\cosh(B/2)}{\sinh(\ell(\varepsilon_0))/4} \right) \quad (3.5)$$

$$\ell(\tau'_0) \leq 4 \sinh^{-1} \left(\frac{\cosh(B/2)}{\sinh(\ell(\varepsilon_0))/4} \right) + B = L_0. \quad (3.6)$$

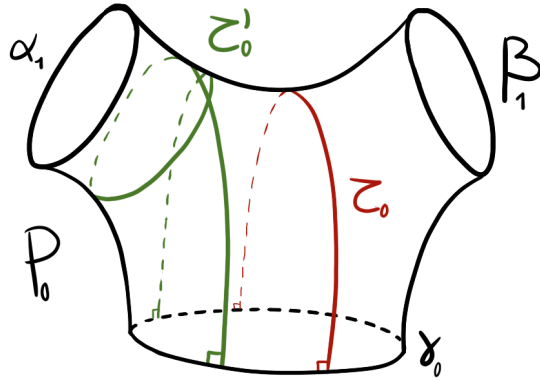


Figure 3.3: The pair of pants P_0 and the two orthogeodesics used to compute a lower bound on $\ell(\alpha_1)$.

By (2.4), we have

$$\ell(\alpha_1) = 2 \cosh^{-1} \left(\frac{\cosh(\ell(\tau'_0)/2)}{\cosh(\ell(\tau_0)/2)} \right).$$

Remark that the function $f(t, d) = 2 \cosh^{-1} \left(\frac{\cosh((t+d)/2)}{\cosh(t/2)} \right)$ is increasing in t and d , thus

$$2 \cosh^{-1} \left(\frac{\cosh((\text{osys}(\chi) + \delta')/2)}{\cosh(\text{osys}(\chi)/2)} \right) \leq 2 \cosh^{-1} \left(\frac{\cosh(\ell_\chi(\tau)/2 + (\ell_\chi(\tau') - \ell_\chi(\tau))/2)}{\cosh(\ell_\chi(\tau)/2)} \right) \quad (3.7)$$

for all orthogeodesics τ and τ' in $\mathcal{O}(\chi)$ and δ' such that $\ell_\chi(\tau) < \ell_\chi(\tau') \leq L$ and

$$\delta' \leq \min\{\ell_\chi(\tau') - \ell_\chi(\tau) \mid \ell_\chi(\tau) < \ell_\chi(\tau') \leq L\}.$$

Thus we have a lower bound ε_1 on the length of α_1 as follow:

$$\varepsilon_1 = 2 \cosh^{-1} \left(\frac{\cosh((\text{osys}(\chi) + \delta(L_0))/2)}{\cosh(\text{osys}(\chi)/(2))} \right) \leq \ell(\alpha_1).$$

By symmetry, ε_1 is also a lower bound on the length of β_1 .

At step $i > 0$, we look at any vertices in T at distance i from the root. Take a path from the root to this vertex, we consider the union of pair of pants that corresponds to vertices on this path, see Figure 3.4. Then we compute an upper bound L_i on the length of two orthogeodesics τ_i and τ'_i as in Figure 3.4.

$$\ell(\tau_i) \leq 2iB + 2 \sinh^{-1} \left(\frac{\cosh(B/2)}{\sinh(\varepsilon_i)/4} \right) + \sum_{j=1}^i \cosh^{-1} \left(\frac{\cosh(B/2)(1 + \cosh(B/2))}{\sinh(\varepsilon_{j-1}/2) \sinh(\varepsilon_j/2)} \right) \quad (3.8)$$

$$\ell(\tau'_i) \leq (2i + 1)B + 4 \sinh^{-1} \left(\frac{\cosh(B/2)}{\sinh(\varepsilon_i)/4} \right) + \sum_{j=1}^i \cosh^{-1} \left(\frac{\cosh(B/2)(1 + \cosh(B/2))}{\sinh(\varepsilon_{j-1}/2) \sinh(\varepsilon_j/2)} \right) = L_i. \quad (3.9)$$

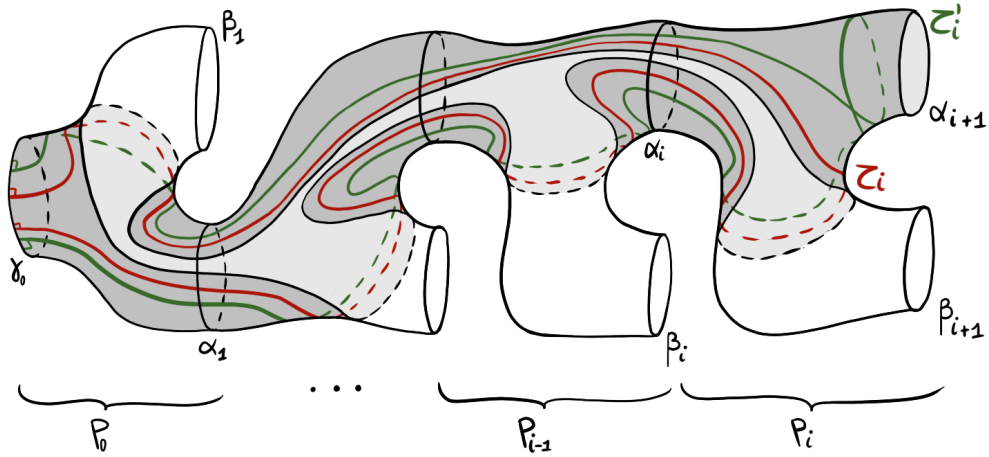


Figure 3.4: Subsurface of Y made out of the different pairs of pants and the orthogeodesics τ_i and τ'_i .

By (2.4), we have

$$\ell(\alpha_{i+1}) = 2 \cosh^{-1} \left(\frac{\cosh(\ell(\tau'_i)/2)}{\cosh(\ell(\tau_i)/2)} \right).$$

Then, by Inequality 3.7

$$\varepsilon_{i+1} = 2 \cosh^{-1} \left(\frac{\cosh((\text{osys}(\chi) + \delta(L_i))/2)}{\cosh(\text{osys}(\chi)/2)} \right) \leq \ell(\alpha_{i+1}).$$

By symmetry, ε_{i+1} is also a lower bound on the length of β_{i+1} .

There is a finite number of vertices in G , so there is only a finite number of steps. By computing a lower bound on the length of the boundary components of each pair of pants corresponding to a vertex in G , we obtain a lower bound on the systole and the length of each boundary components of χ' .

Let n be diameters of T and $L_0(\chi), L_1(\chi), \dots, L_n(\chi) = L(\chi)$ the different upper bounds computed along the induction process for χ .

In the next part we construct q by following each step of the induction process of the previous part of the proof. At each step of this process we also compute upper and lower bounds which will help control the various upper and lower bounds obtained during the induction process for Υ' .

Denote $\mathcal{O}(\chi) \cap [0, L(\chi)] = \{a_1, \dots, a_k\}$ and $a_{k+1} = \min\{a \in \mathcal{O}(\chi) \mid a > a_k\}$. Then define

$$\delta_i = \min_{1 \leq j \leq k-1} \{a_{j+1} - a_j \mid a_{j+1} \neq a_j \text{ and } a_j \leq L_i(\chi)\}.$$

Since $\chi \in \text{Def}(S_g^b) \setminus \bigcup_{[[\tau]] \neq [[\tau']] \mathcal{E}([[\tau]], [[\tau']])}$, each length that appears several times in $\mathcal{O}(\chi)$ is realized by an arc lying in the same equivalence class of $\sim$.

Set $A_q = (24g + 12b - 24) \sinh^{-1} \left(\frac{1}{2 \sinh(2 \text{osys}(\chi)/q)} \right)$, $B_q = \max\{A_q, \text{area}(\chi)\}$ and

$$\delta_q(L) = \min_{1 \leq j \leq k-1} \left\{ \frac{a_{j+1}}{q} - qa_j \mid a_{j+1} \neq a_j \text{ and } a_j \leq L \right\}.$$

We follow the induction process of the previous part of the proof. Let us start with $i = 0$.
Set $\varepsilon_{q_0} = F(q_0 \cdot \text{osys}(\chi))/b$ and

$$L_{q_0} = 4 \sinh^{-1} \left(\frac{\cosh(B_{q_0}/2)}{\sinh(\varepsilon_{q_0})/4} \right) + B_{q_0}$$

Set $a_{j_0} = \min\{a_j \mid a_j > L_0(\chi)\}$ and let $q_0 > 1$ be small enough such that

$$L_{q_0} < \frac{a_{j_0}}{q_0}.$$

Such a constant q_0 does exist: indeed, as q_0 goes to 1, we have $L_{q_0} \rightarrow L_0(\chi)$, $\frac{a_{j_0}}{q_0} \rightarrow a_{j_0}$ and $L_0(\chi) < a_{j_0}$.

Then at $i > 0$, set

$$\varepsilon_{q_i} = 2 \cosh^{-1} \left(\frac{\cosh((\text{osys}(\chi)/q_i + \delta_{q_i}(L_{q_{i-1}}))/2)}{\cosh(\text{osys}(\chi)/(2q_i))} \right)$$

and

$$L_{q_i} = 2i \cdot B_{q_i} + 2 \sinh^{-1} \left(\frac{\cosh(B_{q_i}/2)}{\sinh(\varepsilon_{q_i})/4} \right) + \sum_{l=1}^i \cosh^{-1} \left(\frac{\cosh(B_{q_l}/2)(1 + \cosh(B_{q_l}/2))}{\sinh(\varepsilon_{q_{l-1}}) \sinh(\varepsilon_{q_l})} \right).$$

set $a_{j_i} = \min(a_j \mid a_j > L_i(\chi))$ and let $q_i > 1$ small enough such that

$$L_{q_i} < \frac{a_{j_i}}{q_i}.$$

Such a constant q_i does exist: indeed, as q_i goes to 1 we have $L_{q_i} \rightarrow L_i(\chi)$, $\frac{a_{j_i}}{q_i} \rightarrow a_{j_i}$ and $L_i(\chi) < a_{j_i}$.
At step $i = n$, we have that $L_{q_n} < \frac{a_{k+1}}{q_n}$.

We set

$$q = \min \left\{ q_0, \dots, q_n, \max_{1 \leq j \leq k} \left\{ q' > 1 \mid a_{j+1} - \frac{a_{j+1}}{q'} < \frac{\delta_n}{2} \text{ and } q'a_j - a_j < \frac{\delta_n}{2} \right\} \right\}.$$

Finally, in the last part, let us prove that Proposition 3.3.2 is true for this choice of q . Once again, we follow the steps of the induction process of the first part. For all $\Upsilon \in \mathcal{U}_\chi$ and for all arc or curve β on S_g^b , by Wolpert's Lemma 2.2.16, 2.2.17 we have

$$\frac{\ell_\chi(\beta)}{q} \leq \ell_\Upsilon(\beta) \leq q \ell_\chi(\beta).$$

Set $\{a'_1, \dots, a'_k\} = \mathcal{O}(\Upsilon) \cap [0, \frac{a_{k+1}}{q}] = \mathcal{O}(\Upsilon') \cap [0, \frac{a_{k+1}}{q}]$ and $a'_{k+1} = \min\{a' \in \mathcal{O}(\Upsilon) \mid a' > a'_k\}$. We have $\frac{a_j}{q} \leq a'_j \leq qa_j < \frac{a_{j+1}}{q}$ for all $1 \leq j \leq k$ and $\frac{a_{k+1}}{q} \leq a'_{k+1}$. Moreover, as $\chi \in \text{Def}(S_g^b) \setminus \bigcup_{[[\tau]] \neq [[\tau']] \mathcal{E}([[\tau]], [[\tau']])}$ each length that appears several times in $\mathcal{O}(\chi)$ is realized each time by an

arc of the same class of $\sim$. Thus we have the same for the lengths in $\mathcal{O}(\Upsilon)$ and when $a_j = a_{j+1}$ we also have $a'_j = a'_{j+1}$.

Denote $L_0(\Upsilon'), L_1(\Upsilon'), \dots, L_n(\Upsilon')$ the different upper bounds computed as in (3.6) and (3.9) during the induction process for Υ' , and $\varepsilon_0(\Upsilon'), \dots, \varepsilon_n(\Upsilon')$ the different lower bounds. Since Υ and Υ' have the same orthospectrum, they share the same bounds.

We have an upper bound $A_{\Upsilon'} = (24g + 12b - 24) \sinh^{-1} \left(\frac{1}{2 \sinh(2 \text{osys}(\Upsilon'))} \right)$ on the length of the boundary component of Υ' that is smaller than A_q , thus an upper bound $B_{\Upsilon'} = \max\{A_{\Upsilon'}, \text{area}(\Upsilon')\}$ smaller than B_q . Moreover we have a lower bound $\varepsilon_0(\Upsilon') = F(\text{osys}(\Upsilon'))/b = F(\text{osys}(\Upsilon))/b$ bigger than ε_{q_0} thus $L_0(\Upsilon') < L_{q_0} < \frac{a_{j_0}}{q_0}$.

This implies that

$$\delta_{\Upsilon'}^1 = \min \{ \ell_{\Upsilon'}(\tau') - \ell_{\Upsilon'}(\tau) \mid \ell_{\Upsilon'}(\tau) < \ell_{\Upsilon'}(\tau') \leq L_0(\Upsilon') \} = \min_{1 \leq j \leq k-1} \{ a'_{j+1} - a'_j \mid a_{j+1} \neq a_j \text{ and } a_j \leq L_0(\chi) \}.$$

Thus, $\delta_{\Upsilon'}^1 \geq \delta_{q_1}(L_{q_0}) \geq \delta_q(L_0(\chi))$. Combined with $\frac{\text{osys}(\chi)}{q} \leq \text{osys}(\Upsilon) = \text{osys}(\Upsilon')$, we obtain that $\varepsilon_1(\Upsilon') \geq \varepsilon_{q_1}$.

Then at each step i , we have $\varepsilon_{i-1}(\Upsilon') \geq \varepsilon_{q_{i-1}}$, thus $L_{i-1}(\Upsilon') \leq L_{q_{i-1}} < \frac{a_{j_{i-1}}}{q_{i-1}}$. This implies that

$$\delta_{\Upsilon'}^i = \min \{ \ell_{\Upsilon'}(\tau') - \ell_{\Upsilon'}(\tau) \mid \ell_{\Upsilon'}(\tau) < \ell_{\Upsilon'}(\tau') \leq L_{i-1}(\Upsilon') \} = \min_{1 \leq j \leq k-1} \{ a'_{j+1} - a'_j \mid a_{j+1} \neq a_j \text{ and } a_j \leq L_{i-1}(\chi) \}.$$

Thus $\delta_{\Upsilon'}^i \geq \delta_{q_i}(L_{q_{i-1}}) \geq \delta_q(L_{i-1}(\chi))$ and $\varepsilon_i(\Upsilon') \geq \varepsilon_{q_i}$.

In conclusion, for all $\Upsilon \in \mathcal{U}_\chi$ and $\Upsilon' \in \text{Def}(S_g^b)$ such that $\mathcal{O}(\Upsilon') = \mathcal{O}(\Upsilon)$ we have

$$\text{sys}(\Upsilon') \geq \varepsilon_\chi = 2 \cosh^{-1} \left(\frac{\cosh(\text{osys}(\chi)/q + \delta_q(L(\chi))/2)}{\cosh(\text{osys}(\chi)/(2q))} \right).$$

Moreover, this lower bound on the systole of Υ' is also a lower bound on the length of each boundary component of Υ' . We also have an upper bound $A_\chi = (24g + 12b - 24) \sinh^{-1} \left(\frac{1}{2 \sinh(2 \text{osys}(\chi)/q)} \right)$ on the length of each boundary component. By Corollary 3.1.13, there exists a surface isometric to Υ' in $\mathcal{Q}(A_\chi, \varepsilon_\chi)$. □

In the case of the simple orthospectrum, we have a similar property, this time defined for any open balls in $\text{Def}(S_g^1)$.

Proposition 3.3.3. *Let $\chi \in \text{Def}(S_g^1)$, $q^S > 1$ and $\mathcal{U}_\chi^S$ be an open ball of center χ and radius $\frac{\log(q^S)}{2}$. Then there exist $A_\chi^S, \varepsilon_\chi^S > 0$ such that for all $\Upsilon \in \mathcal{U}_\chi^S$ and $\Upsilon' \in \text{Def}(S_g^1)$ with $\mathcal{O}_S(\Upsilon') = \mathcal{O}_S(\Upsilon)$ there is a surface isometric to Υ' in $\mathcal{Q}(A_\chi^S, \varepsilon_\chi^S)$.*

Proof. Define

$$A_\chi^S = (24g - 12) \sinh^{-1} \left(\frac{1}{2 \sinh(2 \text{osys}(\chi)/q^S)} \right)$$

$$\varepsilon_\chi^S = F(q^S \text{osys}(\chi)).$$

By Wolpert's Lemma 2.2.17, for all $\Upsilon \in \mathcal{U}_\chi^S$ and $\Upsilon' \in \text{Def}(S_g^1)$ such that $\mathcal{O}_S(\Upsilon') = \mathcal{O}_S(\Upsilon)$,

$$\begin{aligned} \ell_\Upsilon(\gamma_0), \ell_{\Upsilon'}(\gamma_0) &\leq A_\chi^S \\ \ell_\Upsilon(\gamma_0), \ell_{\Upsilon'}(\gamma_0) &\geq \varepsilon_\chi^1. \end{aligned}$$

Moreover $\text{sys}(\Upsilon) \geq \frac{\text{sys}(\chi)}{q^S}$ and by definition there are at most $3g + b - 3 = 3g - 2$ curves of length at most ε_m , thus for all $\Upsilon \in \mathcal{U}_\chi^S$, we have by Proposition 3.1.9,

$$\begin{aligned} m(\Upsilon) &\leq c_2(B_\chi^S) \prod_{\gamma: \ell_\Upsilon(\gamma) \leq \varepsilon_m} \left(1 + \frac{1}{\ell_\Upsilon(\gamma) S(\ell_\Upsilon(\gamma))} + \frac{1}{\min\{\ell_\Upsilon(\gamma), S(\ell_\Upsilon(\gamma))\}} \right) \\ &\leq c_2(B_\chi^S) \left(1 + \frac{1}{(\text{sys}(\chi)/q^S) S(\varepsilon_m)} + \frac{1}{\min\{(\text{sys}(\chi)/q^S), S(\varepsilon_m)\}} \right)^{3g-2} \end{aligned}$$

Where $B_\chi^S = \max\{A_\chi^S, 2\pi(2g - 1)\}$. For all $\Upsilon' \in \text{Def}(S_g^1)$ such that $\mathcal{O}_S(\Upsilon') = \mathcal{O}_S(\Upsilon)$, we have $m(\Upsilon') = m(\Upsilon)$, so there is an upper bound on $m(\Upsilon')$ that only depends on χ and q^S . If $\text{sys}(\Upsilon')$ goes to 0 then $m(\Upsilon')$ goes to infinity, which is impossible. So there exists a lower bound ε_χ^2 on $\text{sys}(\Upsilon')$. Set $\varepsilon_\chi^S = \min(\varepsilon_\chi^1, \varepsilon_\chi^2)$. Then by Corollary 3.1.13, there exists a surface isometric to Υ' in $\mathcal{Q}(A_\chi^S, \varepsilon_\chi^S)$. $\square$

Fix a hexagon decomposition $\mathcal{H} = \{\alpha_1, \dots, \alpha_{6g+3b-6}\}$ on S_g^b . Let $\chi_0 \in \text{Def}(S_g^b)$ be given by

$$(\ell_{\chi_0}(\alpha_1), \dots, \ell_{\chi_0}(\alpha_{6g+3b-6})) = (1, \dots, 1).$$

This surface is going to be a point of reference. Recall that, for any $\omega = (\ell_1, \dots, \ell_{6g+3b-6}) \in \mathbb{R}_+^{6g+3b-6}$, we set $\chi^\omega := \varphi_{\mathcal{H}}^{-1}(\omega) \in \text{Def}(S_g^b)$, see Definition 2.2.13. Then we choose a quasi-conformal homeomorphism ϕ^ω between χ_0 and χ^ω . For each orthogeodesic τ on χ_0 , we denote by $\tau(\chi^\omega)$ the unique orthogeodesic in the free homotopy class of the orthogeodesic $\phi^\omega \circ (\tau)$ on χ^ω . For any finite or infinite ordered set $\mathbb{O}$ of orthogeodesics $\tau_1, \tau_2, \dots$ on χ_0 , we define the sequences

$$\begin{aligned} \mathbb{O}(\Upsilon) &= (\tau_1(\Upsilon), \tau_2(\Upsilon), \dots) \\ \ell_\Upsilon(\mathbb{O}) &= (\ell_\Upsilon(\tau_1), \ell_\Upsilon(\tau_2), \dots). \end{aligned}$$

We set $\Pi = (\beta_1, \beta_2, \dots)$ to be the sequence of all orthogeodesics on χ_0 , arranged so that $\ell_{\chi_0}(\beta_1) \leq \ell_{\chi_0}(\beta_2) \leq \dots$ and set $\Pi_k = (\beta_1, \dots, \beta_k)$. Then, set $\Pi_k^S = (\beta_1^S, \beta_2^S, \dots)$ the sequence of all simple orthogeodesics on χ_0 , arranged so that $\ell_{\chi_0}(\beta_1^S) \leq \ell_{\chi_0}(\beta_2^S) \leq \dots$ and set $\Pi_k^S = (\beta_1^S, \dots, \beta_k^S)$. Note that $\ell_{\chi_0}(\Pi) = \mathcal{O}(\chi_0)$ and $\ell_{\chi_0}(\Pi^S) = \mathcal{O}_S(\chi_0)$. Finally, let $\delta_1, \dots, \delta_{6g+3b-6}$ be the unique collection of simple orthogeodesics on χ_0 such that $i(\alpha_i, \delta_j) = \delta_{ij}$ for all $1 \leq i, j \leq 6g + 3b - 6$, and set $\Sigma = \mathcal{H} \cup \{\delta_1, \dots, \delta_{6g+3b-6}\}$.

We fix $\chi \in \text{Def}(S_g^b)$. In the case of the orthospectrum, we assume that $\chi \in \text{Def}(S_g^b) \setminus \bigcup_{[[\tau]] \neq [[\tau']] \mathcal{E}([[\tau]], [[\tau']])}$ but we do not impose such a restriction in the case of the simple orthospectrum except that the surface only has one boundary component. Let $\mathcal{U}_\chi$ and $A_\chi, \varepsilon_\chi > 0$ be as in Proposition 3.3.2. Fix $q_0^S > 1$, let $\mathcal{U}_\chi^S$ be an open ball of center χ and radius $\frac{\log(q_0^S)}{2}$, and $A_\chi^S, \varepsilon_\chi^S > 0$ be as in Proposition 3.3.3. The open neighborhood $\mathcal{U}_\chi$ has a compact closure and there is a compact $\mathcal{Q}(A_\chi, \varepsilon_\chi)$ that contains it. Similarly, the open neighborhood $\mathcal{U}_\chi^S$ has a compact closure and there is a compact $\mathcal{Q}(A_\chi^S, \varepsilon_\chi^S)$ that

contains it.

Let $D = \{\chi^\omega \in \text{Def}(S_g^b) \mid \frac{1}{10} \leq \ell_{\chi^\omega}(\alpha_1), \dots, \ell_{\chi^\omega}(\alpha_{6g+3b-6}) \leq 10\}$. In the case of the orthospectrum, let C, C_1 be compact sets of $\text{Def}(S_g^b)$ whose interiors $\mathring{C}, \mathring{C}_1 \subset \text{Def}(S_g^b)$ are connected and

$$(\mathcal{Q}(A_\chi, \varepsilon_\chi) \cup D) \subset \mathring{C} \subset C \subset \mathring{C}_1.$$

And for the simple orthospectrum, let C^S, C_1^S be compact sets whose interiors $\mathring{C}^S, \mathring{C}_1^S$ are connected and

$$(\mathcal{Q}(A_\chi^S, \varepsilon_\chi^S) \cup D) \subset \mathring{C}^S \subset C^S \subset \mathring{C}_1^S.$$

By Corollary 3.1.14, there exists $q \geq 1$ such that

$$\frac{\ell_\Upsilon(\beta)}{q} \leq \ell_{\Upsilon'}(\beta) \leq q\ell_\Upsilon(\beta) \quad (3.10)$$

for any $\Upsilon, \Upsilon' \in C_1$ and any $\beta \in \Pi$. This q will remain fixed during the proof. We denote q^S the constant obtained for any $\Upsilon, \Upsilon' \in C_1^S$.

Lemma 3.3.4. *For any $k \in \mathbb{N}$, there exists an integer $k^* \geq k$, which depends only on k and C_1 , with the following property. If $\Upsilon', \Upsilon'' \in C_1$ and if $\rho : \Pi_k \rightarrow \Pi$ is an injection such that*

$$\ell_{\Upsilon'}(\Pi_k) = \ell_{\Upsilon''}(\rho(\Pi_k))$$

then $\rho(\Pi_k) \subset \Pi_{k^}$. The same is true with $\Pi^S, \Pi_k^S, \Pi_{k^*}^S$ and C_1^S instead of Π, Π_k, Π_{k^*} and C_1 .*

Proof. Let $c_1 = \max\{\ell_\Upsilon(\beta) \mid \Upsilon \in C_1, \beta \in \Pi_k\}$, and let $k^* \geq k$ be such that, on the base surface χ_0 , we have $\ell_{\chi_0}(\beta_j) > qc_1$ for all $j > k^*$. By (3.10) and since χ_0 is in $D \subset C_1$, we have $\ell_\Upsilon(\beta_j) \geq \frac{\ell_{\chi_0}(\beta_j)}{q} > c_1$ for all $j > k^*$ and $\Upsilon \in C_1$. Since we have $\ell_{\Upsilon''}(\rho(\beta_i)) = \ell_{\Upsilon'}(\beta_i) \leq c_1$ for $i \leq k$ by definition of c_1 , it follows that $\rho(\beta_i) \in \Pi_{k^*}$. For the simple orthospectrum, just replace β_i, C_1, Π, Π_k and Π_{k^*} with $\beta_i^S, C_1^S, \Pi^S, \Pi_k^S$ and $\Pi_{k^*}^S$. $\square$

3.3.2 The first lengths of the orthospectrum

In this section, we will see that any pair Υ, Υ' in a compact subset of the Deformation space have the same orthospectrum, or the same simple orthospectrum if and only if they share a certain finite number of length in their orthospectrum or simple orthospectrum. This is a key step of the proof of Theorem 3.3.1.

We define for each $k \in \mathbb{N}$ the following sets:

$$\begin{aligned} V_k^1 &= \{(\Upsilon, \Upsilon') \in \mathring{C}_1 \times \mathring{C}_1 \mid \ell_\Upsilon(\Pi_k) \subset \mathcal{O}(\Upsilon') \text{ and } \ell_{\Upsilon'}(\Pi_k) \subset \mathcal{O}(\Upsilon)\} \\ V_k &= V_k^1 \cap (\mathring{C} \times \mathring{C}) \\ W_k^1 &= \{(\Upsilon, \Upsilon') \in \mathring{C}_1^S \times \mathring{C}_1^S \mid \ell_\Upsilon(\Pi_k^S) \subset \mathcal{O}_S(\Upsilon') \text{ and } \ell_{\Upsilon'}(\Pi_k^S) \subset \mathcal{O}_S(\Upsilon)\} \\ W_k &= W_k^1 \cap (\mathring{C}^S \times \mathring{C}^S). \end{aligned}$$

Let $k^*, k_S^* \in \mathbb{N}$ be as in Lemma 3.3.4. Given two pairs of injections $\rho_1, \rho_2 : \Pi_k \rightarrow \Pi_{k^*}$ and $\rho_1^S, \rho_2^S : \Pi_k^S \rightarrow \Pi_{k_S^*}^S$, we set

$$V[\rho_1, \rho_2] = \{(\Upsilon, \Upsilon') \in \mathring{C}_1 \times \mathring{C}_1 \mid \ell_\Upsilon(\Pi_k) = \ell_{\Upsilon'}(\rho_1(\Pi_k)) \text{ and } \ell_{\Upsilon'}(\Pi_k) = \ell_\Upsilon(\rho_2(\Pi_k))\}$$

$$W[\rho_1^S, \rho_2^S] = \{(\Upsilon, \Upsilon') \in \mathring{C}_1^S \times \mathring{C}_1^S \mid \ell_\Upsilon(\Pi_k^S) = \ell_{\Upsilon'}(\rho_1^S(\Pi_k^S)) \text{ and } \ell_{\Upsilon'}(\Pi_k^S) = \ell_\Upsilon(\rho_2^S(\Pi_k^S))\}.$$

By Lemma 2.2.14, the spaces $V[\rho_1, \rho_2]$ and $W[\rho_1^S, \rho_2^S]$ are real analytic subvarieties of $\mathring{C}_1 \times \mathring{C}_1$ and $\mathring{C}_1^S \times \mathring{C}_1^S$. Then, thanks to Lemma 3.3.4, we have

$$V_k^1 = \bigcup_{(\rho_1, \rho_2)} V[\rho_1, \rho_2]$$

$$W_k^1 = \bigcup_{(\rho_1^S, \rho_2^S)} W[\rho_1^S, \rho_2^S].$$

Since there are finitely many pairs (ρ_1, ρ_2) and (ρ_1^S, ρ_2^S) , the unions V_k^1 and W_k^1 are also real analytic varieties.

Lemma 3.3.5. *There exists $K \in \mathbb{N}$ such that $V_{K+j} = V_K$ and $W_{K+j} = W_K$ for all $j \geq 1$.*

Proof. The Deformation space is a real analytic space, and C_1 and C_1^S are a compact subsets of $\text{Def}(S_g^b)$. Thus, by [Fri67, Théorème I.9], the rings of real analytic functions on $C_1 \times C_1$ and $C_1^S \times C_1^S$ are Noetherian. Moreover, any real analytic subvariety V is associated with the ideal $I(V)$ of real analytic functions vanishing on the subvariety. For two real analytic subvarieties V_1, V_2 , the inclusion $V_1 \subset V_2$ is equivalent to $I(V_2) \subset I(V_1)$ (see [Bru52] for more details). By definition, increasing sequences of ideals of a Noetherian ring are stationary, so decreasing sequences of subvarieties are stationary. Therefore, there is $K \in \mathbb{N}$ such that $V_{K+j} = V_K$ and $W_{K+j} = W_K$ for all $j \geq 1$. $\square$

3.3.3 Proof of Theorem 3.3.1

We define $\mathcal{U}_\chi \subset \text{Def}(S_g^b)$, D, C and C_1 as in Section 3.3.1. To avoid repetition, we prove Theorem 3.3.1 only for the orthospectrum. For the simple orthospectrum, the proof is the same with $\Pi^S, \mathcal{U}_\chi^S, C^S$ and C_1^S instead of $\Pi, \mathcal{U}_\chi, C$ and C_1 and so on. Note that some verifications we perform about the simplicity of curves are not necessary when proving the theorem for the simple orthospectrum.

Proof of Theorem 3.3.1. We fix K as in Lemma 3.3.5 and large enough so that $\Sigma \subset \Pi_K$. Then, with the notations of Lemma 3.3.4, we fix $M = K^*$ and $N = M^*$. Now, let $\rho : \Pi_M \rightarrow \Pi_N$ be any injection such that $\Pi_K \subset \rho(\Pi_M)$. We define

$$V_\rho = \{\Upsilon \in \mathring{C} \mid \text{there exists } \Upsilon^\rho \in \text{Def}(S_g^b) \text{ such that } \ell_\Upsilon(\Pi_M) = \ell_{\Upsilon^\rho}(\rho(\Pi_M))\},$$

$$\overline{V}_\rho = \{\Upsilon \in \mathring{C} \mid \text{there exists } \Upsilon^\rho \in \mathring{C} \text{ such that } \ell_\Upsilon(\Pi_M) = \ell_{\Upsilon^\rho}(\rho(\Pi_M))\}.$$

For $\Upsilon \in V_\rho$, the surface Υ^ρ is unique: indeed, since $\mathcal{H} \subset \Sigma \subset \Pi_K \subset \rho(\Pi_M)$, the vector $\ell_{\Upsilon^\rho}(\mathcal{H}) = \ell_\Upsilon(\rho^{-1}(\mathcal{H}))$ represents the Ushijima's coordinates of Υ^ρ . Moreover, we have $\overline{V}_\rho \subset V_\rho$, and for $\Upsilon \in \overline{V}_\rho$, we have $\mathcal{O}(\Upsilon) = \mathcal{O}(\Upsilon^\rho)$. We have $\ell_{\Upsilon^\rho}(\Pi_K) \subset \ell_{\Upsilon^\rho}(\rho(\Pi_M)) = \ell_\Upsilon(\Pi_M) \subset \mathcal{O}(\Upsilon)$. Conversely, since $\Pi_K \subset \Pi_M$, we also have $\ell_\Upsilon(\Pi_K) \subset \ell_\Upsilon(\Pi_M) = \ell_{\Upsilon^\rho}(\rho(\Pi_M)) \subset \mathcal{O}(\Upsilon^\rho)$ so $\mathcal{O}(\Upsilon) = \mathcal{O}(\Upsilon^\rho)$

by Lemma 3.3.5.

Now, we define the real analytic mapping $m_\rho : \text{Def}(S_g^b) \rightarrow \text{Def}(S_g^b)$ given by $m_\rho(\chi^\omega) = \chi^{\ell_{\chi^\omega}(\rho^{-1}(\mathcal{H}))}$. If Υ^ρ exists, we have $m_\rho(\Upsilon) = \Upsilon^\rho$. Thus,

$$V_\rho = \{\Upsilon \in \mathring{C} \mid \ell_\Upsilon(\Pi_M) = \ell_{m_\rho(\Upsilon)}(\rho(\Pi_M))\}$$

is a real analytic subvariety of $\mathring{C}$.

We define

$$\begin{aligned} V_\rho^* &= \{\Upsilon \in V_\rho \mid \Upsilon^\rho \text{ is isometric to } \Upsilon\}, \\ \overline{V}_\rho^* &= \{\Upsilon \in \overline{V}_\rho \mid \Upsilon^\rho \text{ is isometric to } \Upsilon\}. \end{aligned}$$

A surface $\Upsilon \in V_\rho$ is isometric to $\Upsilon^\rho = m_\rho(\Upsilon)$ if and only if Υ and Υ^ρ are in the same $\mathcal{MCG}(S_g^b)$ orbit, that is, if and only if there is an injection $r : \mathcal{H} \rightarrow \Pi$ induced by a homeomorphism of the base surface χ_0 such that

$$\ell_\Upsilon(\mathcal{H}) = \ell_{m_\rho(\Upsilon)}(r(\mathcal{H})).$$

By Lemma 3.3.4, we have $r(\mathcal{H}) \subset \Pi_M$. This shows that the set R^* of all such possible injections r is finite. This implies that

$$V_\rho^* = \bigcup_{r \in R^*} \{\Upsilon \in V_\rho \mid \ell_\Upsilon(\mathcal{H}) = \ell_{m_\rho(\Upsilon)}(r(\mathcal{H}))\}$$

is a real analytic subvariety of V_ρ .

Let $\mathcal{R}$ be the set of injective maps $\rho : \Pi_M \rightarrow \Pi_N$ satisfying $\Pi_K \subset \rho(\Pi_M)$. Note that this set is finite. Define

$$\begin{aligned} V &= \bigcup_{\rho \in \mathcal{R}} (V_\rho \setminus V_\rho^*), \\ \overline{V} &= \bigcup_{\rho \in \mathcal{R}} (\overline{V}_\rho \setminus \overline{V}_\rho^*). \end{aligned}$$

The sets V_ρ, V_ρ^* are real analytic subvarieties so V is a real analytic subvariety of $\mathring{C}$. Note that $\overline{V}_\rho^* \subset V_\rho^*$, thus $(\overline{V}_\rho \setminus \overline{V}_\rho^*) \subset (V_\rho \setminus V_\rho^*)$ and $\overline{V}$ is included in a real analytic subvariety of $\mathring{C}$.

By construction, we have $\overline{V} \cap \mathring{C} \subset \mathcal{V}_g^b \cap \mathring{C}$, hence $\overline{V} \cap \mathcal{U}_\chi \subset \mathcal{V}_g^b \cap \mathcal{U}_\chi$. Conversely, if $\Upsilon \in \mathcal{V}_g^b \cap \mathcal{U}_\chi$ then there exists Υ' not isometric to Υ such that $\mathcal{O}(\Upsilon) = \mathcal{O}(\Upsilon')$. By Proposition 3.3.2, we can assume that $\Upsilon' \in \mathcal{Q}(A_\chi, \varepsilon_\chi) \subset \mathring{C}$. Moreover, there exists a bijection $\rho : \Pi \rightarrow \Pi$ satisfying $\ell_\Upsilon(\Pi) = \ell_{\Upsilon'}(\rho(\Pi))$. By Lemma 3.3.4, we have $\rho^{-1}(\Pi_K) \subset \Pi_M$ and $\rho(\Pi_M) \subset \Pi_N$. In other words, there exists an injection $\rho : \Pi_M \rightarrow \Pi_N$ such that

$$\begin{aligned} \Pi_K &\subset \rho(\Pi_M) \\ \ell_\Upsilon(\Pi_M) &= \ell_{\Upsilon'}(\rho(\Pi_M)). \end{aligned}$$

By definition of $\overline{V}$, we have $\Upsilon \in \overline{V} \cap \mathcal{U}_\chi$. In conclusion, $\overline{V} \cap \mathcal{U}_\chi = \mathcal{V}_g^b \cap \mathcal{U}_\chi \subset V \cap \mathcal{U}_\chi$ for any neighborhood $\mathcal{U}_\chi$.

Let us show that $\bar{V} \cap \mathcal{U}_\chi = \mathcal{V}_g^b \cap \mathcal{U}_\chi \neq \mathcal{U}_\chi$. We proceed by showing that $V \cap \mathcal{U}_\chi \neq \mathcal{U}_\chi$, i.e., that $\dim V < \dim \text{Def}(S_g^b)$. Since V_ρ is a real analytic subvariety of $\mathring{C}$ and $\mathring{C}$ is connected by definition, we either have $V_\rho = \mathring{C}$ or else $\dim V_\rho < \dim \mathring{C} = \dim \text{Def}(S_g^b)$. We want to show that if $V_\rho = \mathring{C}$ then $V_\rho = V_\rho^*$. If this is true, for all $\rho \in \mathcal{R}$, $\dim(V_\rho \setminus V_\rho^*) < \dim \text{Def}(S_g^b)$, thus $\dim V < \dim \text{Def}(S_g^b)$.

So suppose $V_\rho = \mathring{C}$, then there is a map $m_\rho : \mathring{C} \rightarrow \text{Def}(S_g^b)$ such that $\ell_{m_\rho(\Upsilon)}(\rho(\Sigma)) = \ell_\Upsilon(\Sigma)$ for all $\Upsilon \in \mathring{C}$. In the following, we abbreviate $\tilde{\beta} := \rho(\beta)$ for any $\beta \in \Sigma$, and $\tilde{\Upsilon} := m(\Upsilon)$ for any $\Upsilon \in \mathring{C}$.

Step 1: $\rho(\mathcal{H})$ is a hexagon decomposition.

Set $\omega = (\frac{1}{4}, \dots, \frac{1}{4})$. Then $\ell_{\tilde{\chi}\omega}(\tilde{\alpha}_i) = \frac{1}{4}$ for $1 \leq i \leq 6g + 3b - 6$. By Theorem 2.3.2, we deduce that the orthogeodesics $\tilde{\alpha}_i$ are simple. Note that in the case of the simple orthospectrum we already know that these orthogeodesics are simple. The fact that they are pairwise disjoint follows from Lemma 2.3.6. If $\tilde{\alpha}_i$ and $\tilde{\alpha}_j$ intersect each other, then $\sinh(\ell_{\tilde{\chi}\omega}(\tilde{\alpha}_i)) \sinh(\ell_{\tilde{\chi}\omega}(\tilde{\alpha}_j)) > 1$. This is impossible since $\sinh(\frac{1}{4}) \sinh(\frac{1}{4}) < 1$. Therefore, ρ sends $\mathcal{H}$ to another hexagon decomposition $\tilde{\mathcal{H}}$ of χ_0 .

Step 2: Understand the relative position of the $\tilde{\alpha}_i$.

We want to show that if $\alpha_{i_1}, \alpha_{i_2}, \alpha_{i_3}, \alpha_{i_4}$ are orthogeodesics delimiting an octagon of orthogonals α_j and δ_j on χ_0 , then $\tilde{\alpha}_{i_1}, \tilde{\alpha}_{i_2}, \tilde{\alpha}_{i_3}, \tilde{\alpha}_{i_4}$ are also orthogeodesics delimiting an octagon of orthogonals $\tilde{\alpha}_j$ and $\tilde{\delta}_j$. Indeed, fix Υ and $1 \leq j \leq 6g + 3b - 6$ such that $\ell_\Upsilon(\alpha_i) = \frac{1}{4}$ and $\ell_\Upsilon(\alpha_j) = 6$ for all $1 \leq i \neq j \leq 6g + 3b - 6$. By Lemma 2.3.9, we have

$$\cosh(\ell_\Upsilon(\delta_j)) = f_{\text{octa}}(\ell_\Upsilon(\alpha_j), \ell_\Upsilon(\alpha_{i_1}), \ell_\Upsilon(\alpha_{i_2}), \ell_\Upsilon(\alpha_{i_3}), \ell_\Upsilon(\alpha_{i_4}))$$

and $\ell_{\tilde{\Upsilon}}(\tilde{\delta}_j) = \ell_\Upsilon(\delta_j) < \frac{1}{2}$. As before, by Theorem 2.3.2 and Lemma 2.3.6, the curve $\tilde{\delta}_j$ is simple and disjoint from $\tilde{\alpha}_i$ for any $1 \leq i \neq j \leq 6g + 3b - 6$. It follows that $i(\tilde{\delta}_j, \tilde{\alpha}_j) = 1$ and the arcs $\tilde{\delta}_j, \tilde{\alpha}_j$ lie in an octagon delimited by four orthogeodesics among the $\tilde{\alpha}_i$ for $i \neq j$. Since

$$\cosh(\ell_{\tilde{\Upsilon}}(\tilde{\delta}_j)) = f_{\text{octa}}(\ell_{\tilde{\Upsilon}}(\tilde{\alpha}_j), \ell_{\tilde{\Upsilon}}(\tilde{\alpha}_{i_1}), \ell_{\tilde{\Upsilon}}(\tilde{\alpha}_{i_2}), \ell_{\tilde{\Upsilon}}(\tilde{\alpha}_{i_3}), \ell_{\tilde{\Upsilon}}(\tilde{\alpha}_{i_4})),$$

we see that if we vary the length of one $\tilde{\alpha}_i \in \tilde{\mathcal{H}}$ and fix the other ones, the length of $\tilde{\delta}_j$ only depends on the lengths of $\tilde{\alpha}_{i_1}, \tilde{\alpha}_{i_2}, \tilde{\alpha}_{i_3}, \tilde{\alpha}_{i_4}$ and $\tilde{\alpha}_j$. This means that the orthogeodesics delimiting the octagon containing $\tilde{\delta}_j$ and $\tilde{\alpha}_j$ are $\tilde{\alpha}_{i_1}, \tilde{\alpha}_{i_2}, \tilde{\alpha}_{i_3}, \tilde{\alpha}_{i_4}$.

Moreover, the non-symmetry of f_{octa} also gives an indication about how the orthogeodesics delimiting the octagon are placed. Indeed, if $\alpha, \delta_1, \delta_2, \delta_3$ and δ_4 are as in Figure 2.3 then we obtain different values of β when we exchange δ_1 and δ_2 or when we exchange δ_1 and δ_4 . Thus, if we choose Υ such that $\alpha_{i_1}, \alpha_{i_2}, \alpha_{i_3}, \alpha_{i_4}, \alpha_j$ and δ_j all have different lengths. Then, we know that if $(\alpha_{i_1}, \alpha_{i_2}, \alpha_j)$ delimits a hexagon, then $(\tilde{\alpha}_{i_1}, \tilde{\alpha}_{i_2}, \tilde{\alpha}_j)$ also delimits a hexagon. Thus, the two surfaces are isometric because they are obtained by gluing isometric hexagons with the same pattern.

In conclusion, if $V_\rho = \mathring{C}$, then $V_\rho = V_\rho^*$ and we have $V_\rho \setminus V_\rho^* = \emptyset$ and $\bar{V} \cap \mathcal{U}_\chi = \mathcal{V}_g^b \cap \mathcal{U}_\chi \neq \mathcal{U}_\chi$.

In the case of the simple orthospectrum, the union of the $\mathcal{U}_\chi^S$ covers the whole Deformation space, thus it is enough to say that $\mathcal{W}_g^b$ is a proper local real analytic subvariety of $\text{Def}(S_g^b)$ thus a negligible set. In the case of the orthospectrum, the union of the $\mathcal{U}_\chi$ does not necessarily cover the whole Deformation space. However, it covers $\text{Def}(S_g^b) \setminus \bigcup_{[[\tau]] \neq [[\tau']] \mathcal{E}([[\tau]], [[\tau']])$, which is dense in the Deformation space, thus $\mathcal{V}_g^b \cap (\text{Def}(S_g^b) \setminus \bigcup_{[[\tau]] \neq [[\tau']] \mathcal{E}([[\tau]], [[\tau']])$ is a local proper real analytic variety and is a negligible set of $\text{Def}(S_g^b)$. By Proposition 3.1.18, $\bigcup_{[[\tau]] \neq [[\tau']] \mathcal{E}([[\tau]], [[\tau']])$ is also a negligible

set of $\text{Def}(S_g^b)$. Thus their union is a negligible set and

$$\mathcal{V}_g^b \subset \left(\mathcal{V}_g^b \cap (\text{Def}(S_g^b) \setminus \bigcup_{[[\tau]] \neq [[\tau']]} \mathcal{E}([[\tau]], [[\tau']])) \right) \bigcup_{[[\tau]] \neq [[\tau']]} \mathcal{E}([[\tau]], [[\tau']]).$$

is included in a negligible set of $\text{Def}(S_g^b)$. □

3.4 Example of the one-holed torus

In [MM23], Masai and McShane proved orthospectrum rigidity for the one-holed torus. In the case of the simple orthospectrum, the same proof does not work as it relies on computing the length of the boundary using Basmajian's identity. However in this section, we prove simple orthospectrum rigidity for the one-holed torus with a different proof, which relies on Ushijima's coordinates instead of Fenchel-Nielsen coordinates.

The first result we need to prove rigidity is the following:

Proposition 3.4.1. *Let X be a compact hyperbolic surface with geodesic boundary. Then the first two lengths in $\mathcal{O}_S(X)$ are the lengths of two disjoint orthogeodesics.*

Proof. Let τ_1 and τ_2 be two orthogeodesics realizing the first two lengths of the simple orthospectrum with $\ell(\tau_1) \leq \ell(\tau_2)$. Let us suppose that $i(\tau_1, \tau_2) > 0$. The idea is to get a contradiction by constructing a new orthogeodesic τ shorter than τ_2 .

We construct a piecewise geodesic path σ as follows. Start at an endpoint of τ_1 such that the length between this endpoint and the first intersection point p between τ_1 and τ_2 is less than $\frac{\ell(\tau_1)}{2}$. Then follow τ_1 until p , and finally follow τ_2 until its closest endpoint. We obtain

$$\ell(\sigma) \leq \frac{\ell(\tau_2)}{2} + \frac{\ell(\tau_1)}{2} \leq \ell(\tau_2).$$

Note that σ is essential, otherwise, together with an arc of the boundary of X , we get a hyperbolic triangle with two right angles, which is impossible.

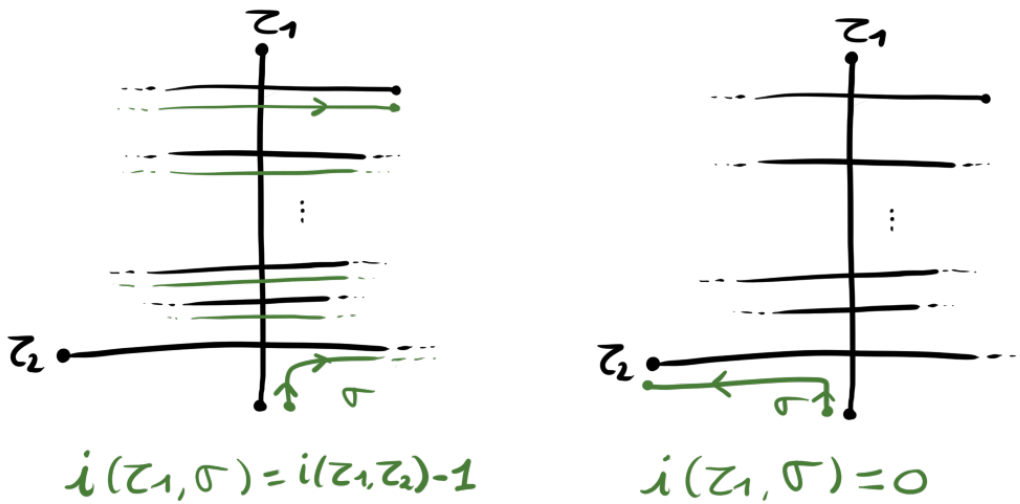


Figure 3.5: Construction of σ .

Note that the orthogeodesic τ homotopic to σ is simple. Since the two arcs forming σ meet at some angle different from π , the geodesic representative τ is strictly shorter than σ and $\ell(\tau) < \ell(\sigma) \leq \ell(\tau_2)$. Moreover, by construction $i(\tau, \tau_1) < i(\tau_1, \tau_2)$. Suppose then that $\tau = \tau_1$. In this case, the segment of τ_2 that σ follows and the segment of τ_1 that σ does not follow are homotopic and form with a segment of ∂X a hyperbolic triangle with two right angles. This is impossible, so $\tau \neq \tau_1$.

Thus, the simple orthogeodesic τ is different from τ_1 and shorter than τ_2 , which is a contradiction. $\square$

With this result at hand, we can prove the desired rigidity statement.

Theorem 3.4.2. *Let T and T' be two hyperbolic structures with geodesic boundary on the one-holed torus. Then T and T' are isometric if and only if $\mathcal{O}_S(T) = \mathcal{O}_S(T')$.*

Proof. A hexagon decomposition of a one-holed torus is formed by three arcs. Our goal is to find a hexagon decomposition where the three orthogeodesics have length $t_1 \leq t_2 \leq t_3$, which are the first three lengths of the simple orthospectrum.

By Proposition 3.4.1, the first two lengths t_1 and t_2 of $\mathcal{O}_S(T) = \mathcal{O}_S(T')$ correspond to two disjoint simple orthogeodesics τ_1 and τ_2 on T (respectively τ'_1 and τ'_2 on T'). To visualize this situation, we give τ_1 and τ_2 an orientation, and cut the one-holed torus along τ_1 and τ_2 to obtain a right-angled octagon as in Figure 3.6.

There are exactly two simple orthogeodesics, τ_3 and $\tilde{\tau}_3$, disjoint from τ_1 and τ_2 , each of which joins two opposite sides of the octagon corresponding to arcs in ∂X . Assume that $\ell(\tau_3) \leq \ell(\tilde{\tau}_3)$. Our goal is to prove that any simple orthogeodesic which intersects τ_1 or τ_2 (or both) is longer than τ_3 .

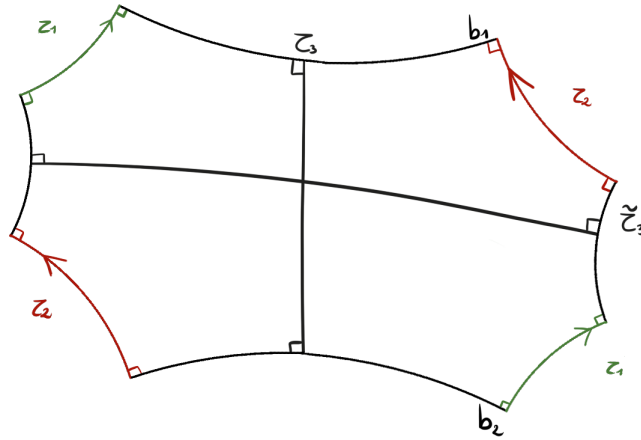


Figure 3.6: τ_3 and $\tilde{\tau}_3$ on the octagon

Let τ be a simple orthogeodesic intersecting τ_1 or τ_2 (or both). Without loss of generality, we can assume that τ has its endpoints on the same sides b_1 and b_2 as τ_3 . Indeed, if τ has its endpoints on the opposite sides, we just replace τ_3 by $\tilde{\tau}_3$ in the proof and show that $\ell(\tau_3) \leq \ell(\tilde{\tau}_3) < \ell(\tau)$. Orient τ from its endpoint on b_1 to the endpoint on b_2 . Assume that τ first intersects τ_1 before possibly intersecting τ_2 (the other case being analogous). Let n be the number of time that τ intersects τ_1 before possibly intersecting τ_2 . We prove that $\ell(\tau_3) < \ell(\tau)$ by induction on n .

Base case $n = 1$: The orthogeodesic τ intersects τ_1 exactly once before intersecting τ_2 .

We denote by p_1 the first point of intersection between τ and τ_1 , and by p_2 the last one. Since τ is simple, p_2 does not lie between p_1 and b_2 . In other words, we have $d(p_2, b_1) + d(p_1, b_2) \leq \ell(\tau_1)$. We label by x the segment of τ between p_2 and b_2 , and by y the segment of τ between p_1 and b_1 (as in Figure 3.7).

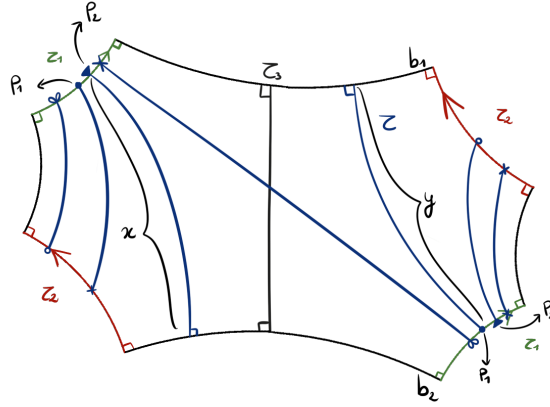


Figure 3.7: Example of case $n = 1$.

We construct two arcs σ and $\tilde{\sigma}$ homotopic to τ_3 as follows: σ is the union of x with the segment of τ_1 between p_2 and b_1 , and $\tilde{\sigma}$ is the union of y with the segment of τ_1 between p_1 and b_2 , as in Figure 3.8. We have

$$\begin{aligned} \text{length}(\sigma) &= d(p_2, b_1) + \text{length}(x) \\ \text{length}(\tilde{\sigma}) &= d(p_1, b_2) + \text{length}(y). \end{aligned}$$

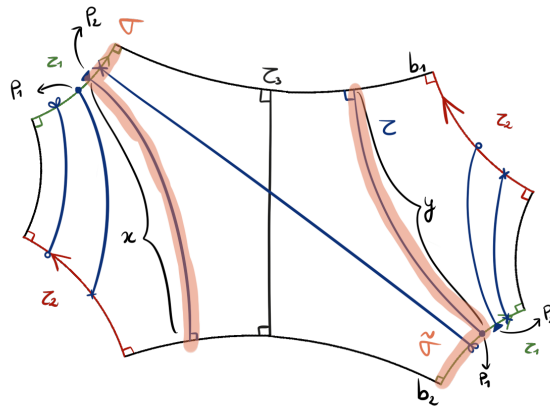


Figure 3.8: Construction of σ and $\tilde{\sigma}$ for case $n = 1$.

We obtain

$$2\ell(\tau_3) < \text{length}(\sigma) + \text{length}(\tilde{\sigma}) \leq d(p_2, b_1) + d(p_1, b_2) + \text{length}(x) + \text{length}(y) \leq \ell(\tau_1) + \ell(\tau).$$

Since $\ell(\tau_1) \leq \ell(\tau)$, we conclude that $\ell(\tau_3) < \ell(\tau)$.

Induction step: Suppose the result is true for any simple orthogeodesic which intersects τ_1 at most n times before intersecting τ_2 . Let τ be a simple orthogeodesic which intersects τ_1 $n+1$ times before intersecting τ_2 .

We denote by $p_1, p_2, \dots, p_n, p_{n+1}$ the first $n+1$ intersection points of τ and τ_1 . These points cut τ_1 into $n+2$ segments.

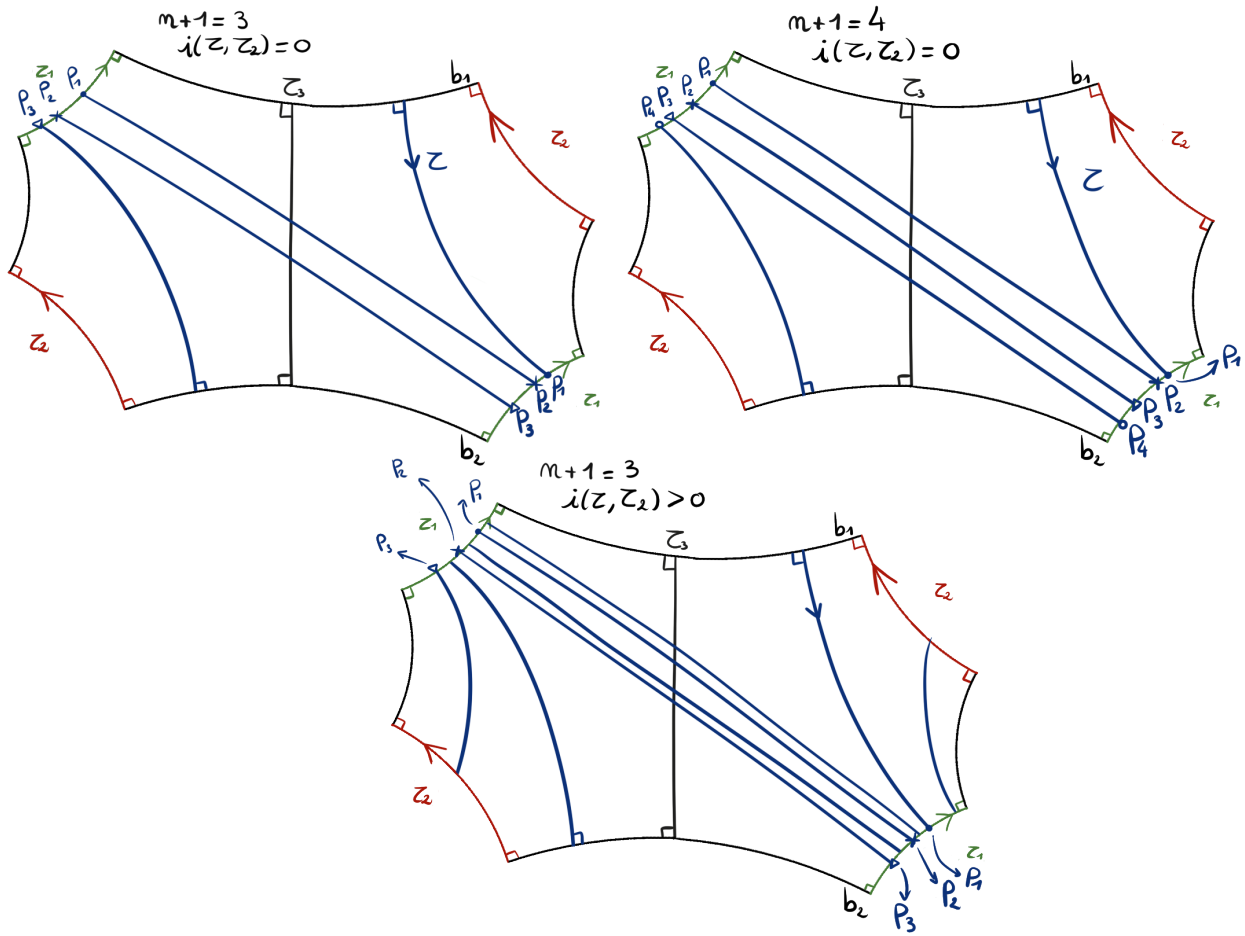


Figure 3.9: Several example of τ .

Let us construct two arcs σ and $\tilde{\sigma}$ as follows. For σ , we start from the edge b_2 and we follow τ until it intersects τ_1 , then we follow τ_1 until p_n , then we follow τ between p_n and p_{n-1} , then τ_1 between p_{n-1} and p_{n-2} and so on until we reach p_1 . If $n+1$ is even, we close up σ by following τ until the edge b_1 ; if $n+1$ is odd, we follow τ_1 until the edge b_1 . We construct $\tilde{\sigma}$ in a similar way, using

segments of τ and τ_1 between the intersection points p_i that we did not already use. We start from the side b_2 and follow τ_1 until p_{n+1} , then we follow τ until p_n , then τ_1 until p_{n-1} and we repeat the process until we reach p_1 . Then, if $n+1$ is even, we follow τ_1 until b_1 ; if $n+1$ is odd, we follow τ until b_1 .

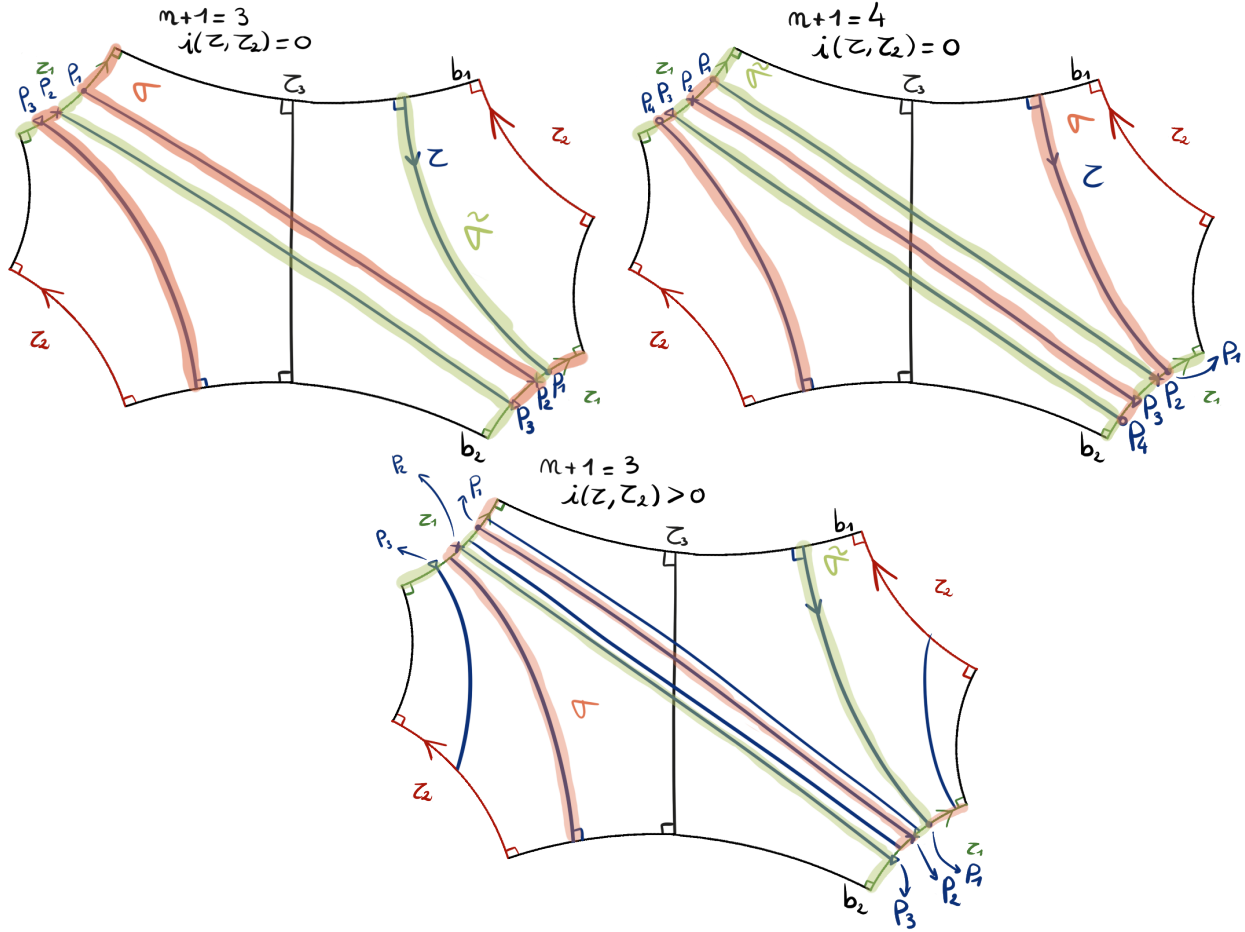


Figure 3.10: Several examples of the construction of σ and $\tilde{\sigma}$ in Case 2.

Since σ and $\tilde{\sigma}$ use different segments of τ and τ_1 , we have $\ell(\sigma) + \ell(\tilde{\sigma}) \leq \ell(\tau_1) + \ell(\tau)$. By induction, we know that $\ell(\tau_3) < \ell(\sigma)$ and $\ell(\tau_3) < \ell(\tilde{\sigma})$. Indeed, when $n+1$ is even, the arc σ is homotopic to τ in the induction step $\frac{n+1}{2}$ for $i(\tau, \tau_2) = 0$ and the arc $\tilde{\sigma}$ is homotopic to τ in the induction step $(\frac{n+1}{2} - 1)$ for $i(\tau, \tau_2) = 0$. When $n+1$ is odd, the arcs σ and $\tilde{\sigma}$ are homotopic to τ in the induction step $\frac{n}{2}$ for $i(\tau, \tau_2) = 0$. To conclude, we have $2\ell(\tau_3) < \ell(\sigma) + \ell(\tilde{\sigma}) \leq \ell(\tau) + \ell(\tau_1)$. Thus, $\ell(\tau_3) < \ell(\tau)$.

So the first three lengths of $\mathcal{O}_S(T)$ and $\mathcal{O}_S(T')$ are realized by disjoint orthogeodesics τ_1, τ_2 and τ_3 on T and τ'_1, τ'_2 and τ'_3 on T' . The set $\{\tau_1, \tau_2, \tau_3\}$ is a hexagon decomposition of T and the set $\{\tau'_1, \tau'_2, \tau'_3\}$ is also a hexagon decomposition of T' . Cutting T and T' along their respective hexagon decomposition, we obtain two isometric sets of two hexagons. We have only two hexagons per set and they are isometric, so there is no ambiguity as to how to glue them back into T and T' , which are then isometric. $\square$

First kind surfaces

In this chapter we mainly focus on one kind of infinite-type surface: the tight flute surface. This work is the result of a collaboration with Gordillo Herrerías [GL25]

Definition 4.0.1. A *flute surface* is an infinite-type surface with genus zero where the space of ends $E(X)$ is homeomorphic to $(1/n)_{n \in \mathbb{N}} \cup \{0\}$.

A flute surface is called *tight* if it is endowed with a hyperbolic metric where every isolated end is a cusp. To geometrically identify the different tight flute surfaces we are working with, we fix a pants decomposition as follows. Let $\mathcal{P} = \{P_1, \dots, P_n, \dots\}$ be a pants decomposition of the tight flute surface such that P_1 is a pair of pants with two cusps and one boundary component α_1 , and for $n > 1$, P_n is a pair of pants with one cusp and two boundary components α_{n-1} and α_n . We then use Fenchel-Nielsen coordinates along this pants decomposition to denote any tight flute surface X : $X = \{\ell_n, t_n\}_{n \in \mathbb{N}^*}$, where ℓ_n is the length of α_n and t_n its twist. We assume t_n to be in $] -\frac{1}{2}, \frac{1}{2}]$ as any other tight flute surface would be isometric to a tight flute surface with its twist parameters in $] -\frac{1}{2}, \frac{1}{2}]$. Moreover, we orient each α_n such that α_{n+1} is to the right of α_n , see Figure 4.1.

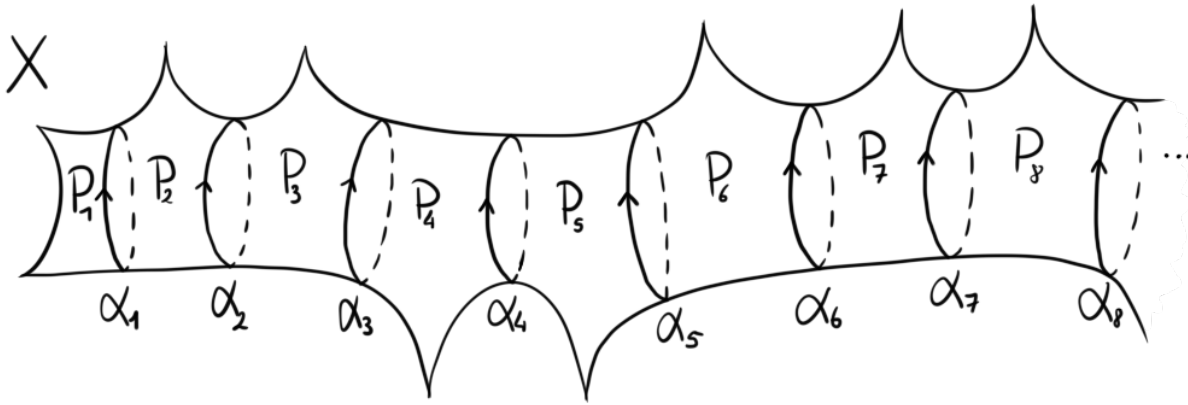


Figure 4.1: Orientation of the α_n in X .

Then let us further cut each pair of pants P_n for $n > 1$ into two isometric pentagons along the simple orthogeodesic connecting the two boundary components and the two orthorays based at each boundary components and going into the cusp. To differentiate the two pentagons in each pair of pants we use the following definitions. In P_n , the *upward pentagon* P_n^1 is the pentagon whose sides inside α_{n-1} and α_n are going from the base of the pentagon toward an infinite side of the pentagon, in the same orientation as α_{n-1} and α_n . The other pentagon is called the *downward pentagon* and we denote it by P_n^{-1} . See Figure 4.2.

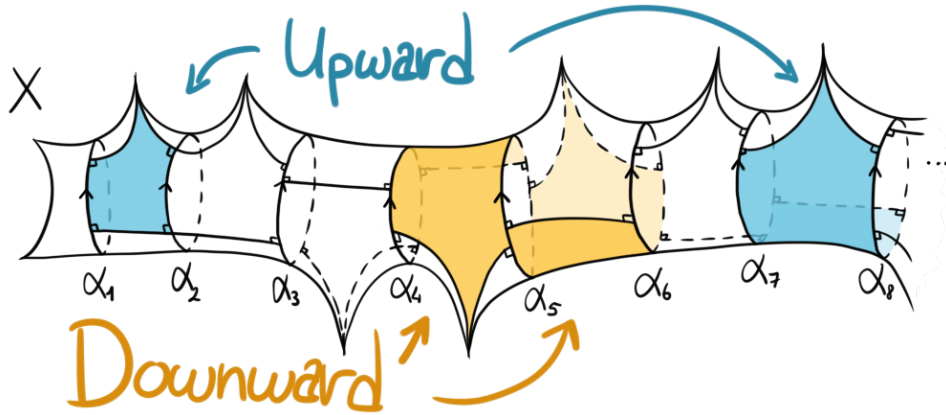


Figure 4.2: Examples of upward and downward pentagons.

This framework will be used all along the chapter. Note that if we replace each cusp by a boundary, we can still define the pair of pants decomposition in the same way. When cutting the pairs of pants, the pentagons will become hexagons, but we can still define upward and downward hexagons in the same way.

4.1 Visible ends

Although we are focusing on flute surfaces, this section states definitions and properties valid for any hyperbolic surface. Let us recall Definition 2.1.5:

Definition 4.1.1. A Fuchsian group Γ (and by extension a hyperbolic surface of Fuchsian group Γ) is of the *first kind* if $\Lambda(\Gamma) = \partial\mathbb{H}$. Otherwise, it is of the *second kind*.

In the case of finite-type surfaces, it is quite easy to understand if a hyperbolic surface is of the first kind: if the surface has at least one boundary component or a funnel it is of the second kind, otherwise it is of the first kind. However, in the case of infinite-type surfaces it is a bit more complicated as infinite-type ends comes into play. In this chapter we wonder under which conditions on the Fenchel-Nielsen coordinates can we tell if an infinite-type surface is of the first kind.

On top of the different topological characterizations of an end that were discussed in Chapter 2, an end can also be visible or non-visible, a notion introduced by Basmajian and Šarić in [BŠ19].

Definition 4.1.2. Let S be a surface and $E(S)$ its space of ends. We say that a path $\gamma : [0, \infty) \rightarrow X$ exits an end $e \in E(X)$ if $\gamma(t)$ converges to e in $X \cup E(X)$.

We will use a definition of visibility which differs from what is given by Basmajian and Šarić, but is equivalent to it by [BŠ19]. Recall that a *geodesic ray* on a surface X is a one-way infinite geodesic.

Definition 4.1.3. Let e be an end of a hyperbolic surface X . Then e is *visible* if there exists a geodesic ray that exits e and leaves the convex core $C(X)$ in finite time. Otherwise e is *non-visible*.

In other words, an end is visible if and only if there exists a geodesic ray that exits e and has finite length in the convex core. There are alternative ways to check if an infinite-type end is visible or not [BŠ19, Lemma 5.3]:

Lemma 4.1.4. *Let X' be a (not necessarily complete) hyperbolic surface constructed by gluing pairs of pants, and let X be its geodesic completion as in Theorem 2.1.31. Fix a simple closed geodesic α on the hyperbolic surface X' , and let e be an infinite-type end of X . The following are equivalent:*

1. *e is a visible end.*
2. *There exists an orthoray based at α that exits the end e and eventually leaves $C(X)$.*
3. *There exists an interval $I \subset \alpha$ such that each orthoray based in I exits the end e and eventually leaves $C(X)$.*

The notion of visible end is key to determine if a surface is of the first kind, as shown by the following result [BŠ19, Proposition 3.7].

Proposition 4.1.5. *Let X' be a (not necessarily complete) hyperbolic surface constructed by gluing pairs of pants, and let X be its geodesic completion as in Theorem 2.1.31. Then:*

1. *The closure of X' in X is $C(X)$. In particular, X is of the first kind if and only if $X' = X$.*
2. *The surface X (and thus X') is of the first kind if and only if every end of X is not visible.*

4.2 First kind characterization of flute surfaces

The goal of this section is to establish which conditions on the Fenchel-Nielsen coordinates of a tight flute surface ensure that X is of the first kind. As we will use Lemma 4.1.4, let us first determine when an orthoray based at α_1 and exiting the infinite-type end of X leaves $C(X)$. To do that, let us introduce the concept of patchwork.

Let σ be an orthoray based at α_1 that exits the infinite-type end of X . We do not consider orthorays that are going back into pairs of pants they already visited, as they are longer than orthorays that do not, and we do not need these longer ones to use Lemma 4.1.4. Similarly, we assume that in each pair of pants σ goes from the closure of one pentagon to the closure of the other at most once, otherwise we can map part of the orthoray into one of the pentagons using the orientation-reversing isometry between them and then pull tight to get a shorter orthoray with the desired property, see Figure 4.3.

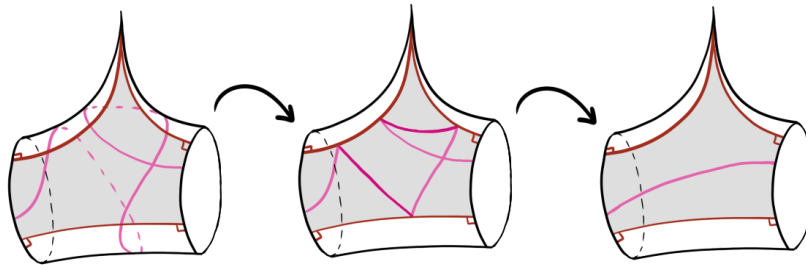
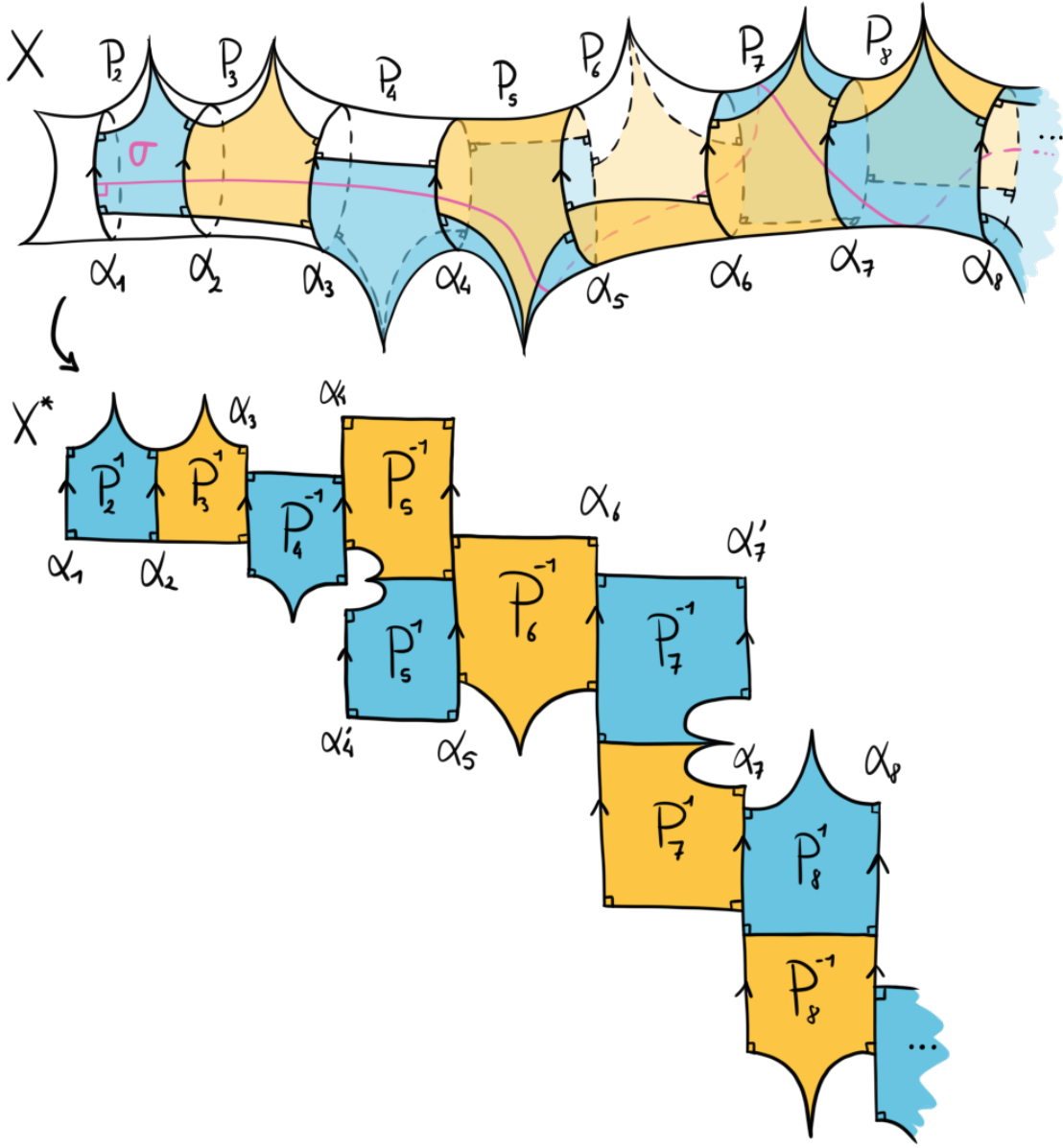


Figure 4.3: The orthoray goes from one pentagon of a pair of pants to the other at most once.

Then, the patchwork X^* is the disjoint union of all pentagons that contains part of σ that we then reconnect along all the sides and vertices of pentagons that σ intersects. By construction X^* is simply connected, see Figure 4.4.

Figure 4.4: Example of a patchwork containing an orthoray σ .

To describe the patchwork associated to an orthoray we use two sequences. We define $(v_n)_{n \in \mathbb{N}^*} \in \{-1, 1\}^{\mathbb{N}^*}$ such that σ enters the pair of pants P_n through $P_n^{v_{2n-1}}$ and leaves it through $P_n^{v_{2n}}$. This sequence describes the direction (upward or downward) of each pentagon through which the orthoray σ enters and exits a pair of pants.

Then we define $(w_n)_{n \in \mathbb{N}^*}$ such that

$$w_{2n-1} = \begin{cases} 1 & \text{if } \sigma \text{ intersects the orthogonal to } \alpha_{n-1} \text{ with the cusp of } P_n \text{ as a vertex} \\ 0 & \text{otherwise.} \end{cases}$$

$$w_{2n} = \begin{cases} 1 & \text{if } \sigma \text{ intersects the orthogonal to } \alpha_n \text{ with the cusp of } P_n \text{ as a vertex} \\ 0 & \text{otherwise.} \end{cases}$$

If the orthoray σ visits both pentagons of a pair of pants, this sequence describes where the orthoray crosses a boundary of the pentagon in the pair of pants. Moreover, because the orthoray only goes from one pentagon of a pair of pants to the other at most once, w_{2n-1} and w_{2n} cannot be both equal to 1 at the same time.

We now properly define a patchwork as follows:

Definition 4.2.1. A *patchwork* is a pair of sequences $((v_n)_{n \in \mathbb{N}^*}, (w_n)_{n \in \mathbb{N}^*}) \in \{-1, 1\}^{\mathbb{N}^*} \times \{0, 1\}^{\mathbb{N}^*}$ such that for $n > 1$, we have $w_{2n-1} + w_{2n} \neq 2$.

Theorem 4.2.2. Let $X = \{\ell_n, t_n\}_{n \in \mathbb{N}^*}$ be a tight flute surface and let σ be an orthoray based at α_1 with $((v_n)_{n \in \mathbb{N}^*}, (w_n)_{n \in \mathbb{N}^*})$ its associated patchwork and such that $w_{2n+2} + w_{2n+3} \neq 2$. Then, the sequence $(u_n)_{n \in \mathbb{N}}$ defined as

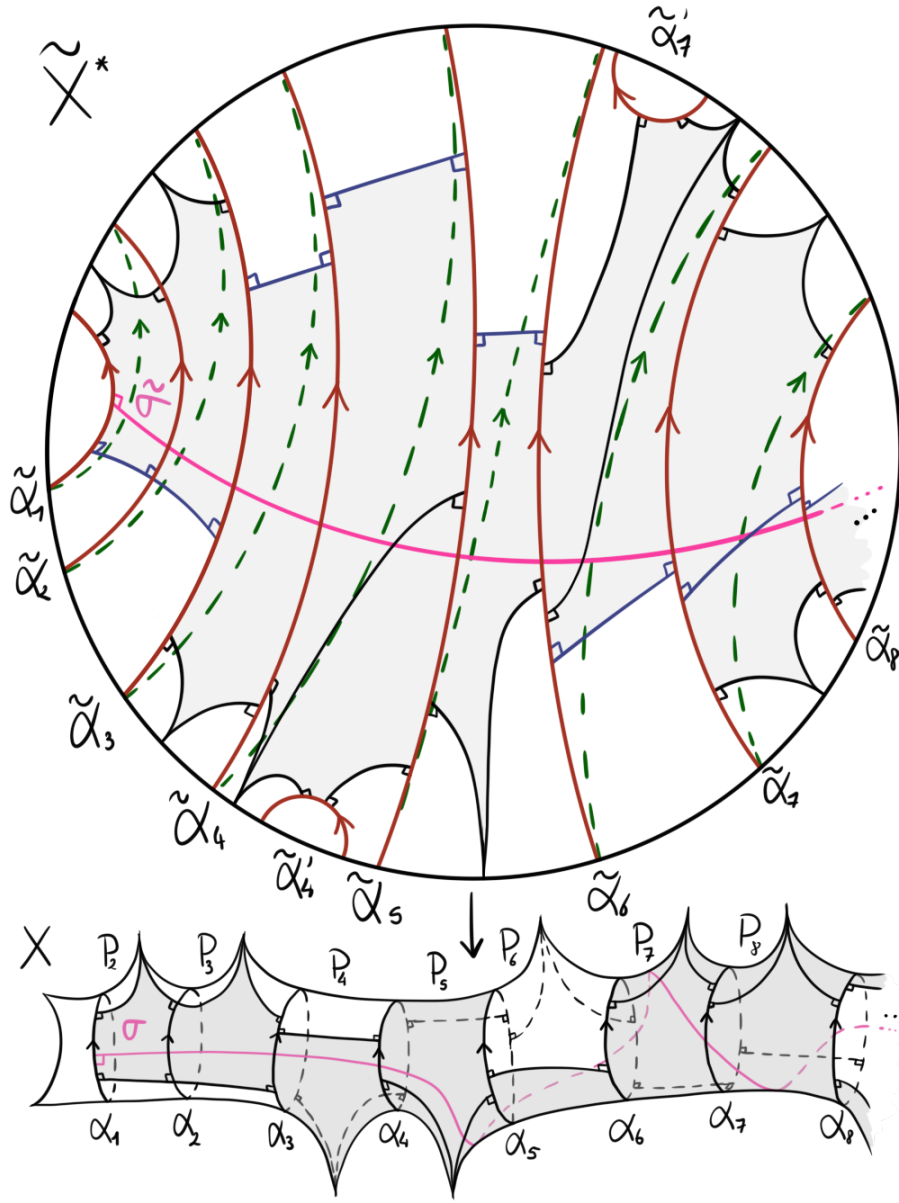
$$u_n = \begin{cases} t_n + (w_{2n} + w_{2n+1})v_{2n-1}(1 - 2w_{2n-1}) & \text{if } v_{2n}v_{2n+1} = 1 \\ t_n & \text{if } v_{2n}v_{2n+1} = -1 \text{ and } v_{2n}(1 - 2(w_{2n} + w_{2n+1}))t_n > 0 \\ t_n + v_{2n}(1 - 2(w_{2n} + w_{2n+1})) & \text{otherwise} \end{cases}$$

is such that the length of σ in $C(X)$ is infinite if and only if

$$\sum_{n=1}^{\infty} \left(e^{-\frac{\ell_{n+1}}{2}} + e^{-\frac{\ell_n}{2}} \right) \cosh(u_n \ell_n + \dots + u_1 \ell_1) = \infty.$$

In other words, an orthoray based at α_1 that exits the infinite-type end of a flute surface eventually leaves the convex core of the flute if and only if its associated sum goes to infinity.

Proof. The patchwork $((v_n)_{n \in \mathbb{N}^*}, (w_n)_{n \in \mathbb{N}^*})$ describes a simply connected surface X^* that can be immersed in X . We fix an isometric lift $\tilde{X}^*$ of X^* to the universal covering $\mathbb{H}$. The lift $\tilde{X}^*$ is a polygon with infinitely many sides; each vertex of $\tilde{X}^*$ on $\partial\mathbb{H}$ corresponds to a cusp of X (see Figure 4.5). Beside its ideal vertices, the closure of the infinite polygon $\tilde{X}^*$ in $\mathbb{H} \subset \partial\mathbb{H}$ contain a single point in $\partial\mathbb{H}$ or two points in $\partial\mathbb{H}$. Let $\tilde{\sigma}$ be the lift of the orthoray σ inside $\tilde{X}^*$, as the lift of X^* is isometric to it, σ and $\tilde{\sigma}$ have the same length.

Figure 4.5: Lift $\tilde{X}^*$ of X^* in the universal covering $\mathbb{H}$.

Denote by g_{2n-1} the geodesic of $\mathbb{H}$ containing the lift of α_n intersected by $\tilde{\sigma}$, as in Figure 4.5. If the vertices of the infinite polygon accumulate to two points in $\partial\mathbb{H}$, the g_{2n-1} accumulate to a geodesic with these two points as endpoints, thus $\ell(\sigma)$ is finite in $C(X)$. Otherwise the g_{2n-1} accumulate to one point and $\ell(\sigma) = \infty$ in $C(X)$.

Orient g_{2n-1} coherently with the orientation of the α_n , with this orientation, g_{2n+1} is to the right of g_{2n-1} . Let g_{2n} be the geodesic which starts at the initial point of g_{2n-1} and finishes at the terminal point of g_{2n+1} . Denote by η_n the common orthogonal between g_{2n-1} and g_{2n+1} . For $n > 1$, let $s_n = s(g_n)$ be the shear of the two ideal triangles in the quadrilateral of diagonal g_n whose vertices are the endpoints of g_{n-1} and g_{n+1} .

To show that the ideal vertices of $\tilde{X}^*$ accumulate to a single point, we show that the sequence of nested geodesics $(g_n)_{n=1}^\infty$ does not accumulate in $\mathbb{H}$. If $\sum \ell(\eta_n) = \infty$ then $(g_n)_{n=1}^\infty$ cannot accumulate in $\mathbb{H}$ (and the sum in the statement of the theorem is necessarily infinite as the hyperbolic cosine function is greater than one). Now suppose that $\sum \ell(\eta_n) < \infty$. This implies that

$$1 \leq \prod_{n=1}^{\infty} (1 + \ell(\eta_n)) < \infty. \quad (4.1)$$

This assumption also means that for n large enough, $\ell(\eta_n)$ is arbitrarily small.

The space between g_n and g_{n+1} is called a *wedge*, and its vertex is the common endpoint of g_n and g_{n+1} . Each wedge is foliated by horocyclic arcs orthogonal to its sides and lying on horocycles whose center is the vertex of the corresponding wedge. Fix a point H_1 on g_1 such that the horocyclic arc starting on H_1 and connecting g_1 and g_2 has length e^{-s_1} . There exists a unique piecewise horocyclic path h starting from H_1 that consists of horocyclic arcs connecting the sides $(g_n)_{n=1}^\infty$ of the wedges. By [PŠ23, Proposition A.1], the sequence $(g_n)_{n=1}^\infty$ does not accumulate in $\mathbb{H}$ if and only if the piecewise horocyclic path h connecting the adjacent geodesic has infinite length. Our goal is to relate the length of h with an explicit sum depending on the Fenchel-Nielsen coordinates of the surface.

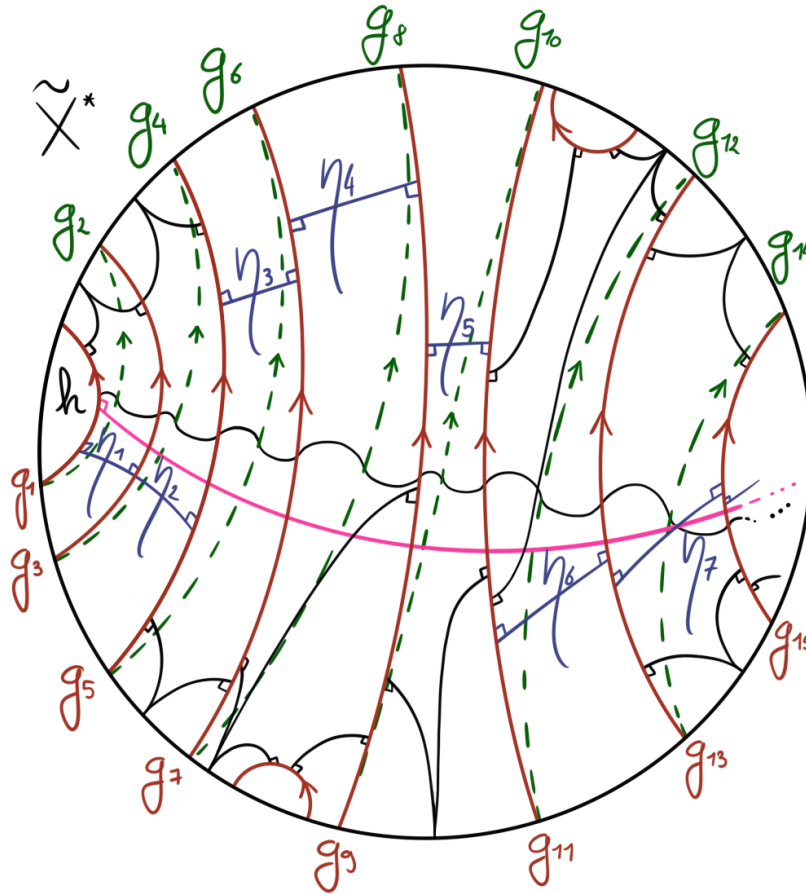


Figure 4.6: Piecewise horocyclic path h on the lift $\tilde{X}^*$ of X^* in the universal covering $\mathbb{H}$.

By [PŠ23, Proposition A.3], the length of the part of the piecewise horocyclic path between g_n and g_{n+1} is $e^{-s_1-s_2-\dots-s_n}$ when n is odd, and $e^{s_1+s_2+\dots+s_n}$ when n is even. Then

$$\ell(h) = \sum_{n=1}^{\infty} e^{s_{2n}+s_{2n-1}+\dots+s_1} + \sum_{n=1}^{\infty} e^{-s_{2n+1}-s_{2n}-\dots-s_1}.$$

We divide the remainder of the proof into two steps: first, computing the shears s_n ; second, obtaining upper and lower bounds for the length of h in terms of

$$\sum_{n=1}^{\infty} \left(e^{-\frac{\ell_{n+1}}{2}} + e^{-\frac{\ell_n}{2}} \right) \cosh(u_n \ell_n + \dots + u_1 \ell_1).$$

Step 1

When n is even, by [PŠ23, Lemma 5.2] we have

$$s_{2n} = s(g_{2n}) = 2 \log \sinh \left(\frac{\ell(\eta_n)}{2} \right). \quad (4.2)$$

Let us compute the shear when n is odd. There are many possible configurations depending on the patchwork, that is, depending on if there are one or two pentagons on the left and right of g_{2n+1} , if they are upward or downward and in which order, their twist and if there are two pentagons one the left or right of g_{2n+1} , depending on how $\tilde{\sigma}$ goes through them. Luckily some configurations can be ignored and some are easier to handle.

In all cases denote by A the starting point of g_{2n} and D the end point of g_{2n+2} . Then denote by P the foot of the orthogeodesic from A on g_{2n+1} and S the foot of the orthogeodesic from D on g_{2n+1} . Let Q be the endpoint of η_n on g_{2n+1} and R the endpoint of η_{n+1} on g_{2n+1} . By (2.3.7) applied to $ABQP$ and $RSDC$, we obtain $\ell(PQ) = \sinh^{-1} \left(\frac{1}{\sinh \ell(\eta_n)} \right)$ and $\ell(RS) = \sinh^{-1} \left(\frac{1}{\sinh \ell(\eta_{n+1})} \right)$. See Figures 4.7, 4.8, 4.10, 4.11, 4.12, 4.13, 4.14, 4.16, 4.18 and 4.19.

Let us start with the computation for configurations for which $v_{2n+1} = v_{2n+2}$ and $v_{2n+3} = v_{2n+4}$ (which implies that $w_{2n+1} = w_{2n+2} = w_{2n+3} = w_{2n+4} = 0$).

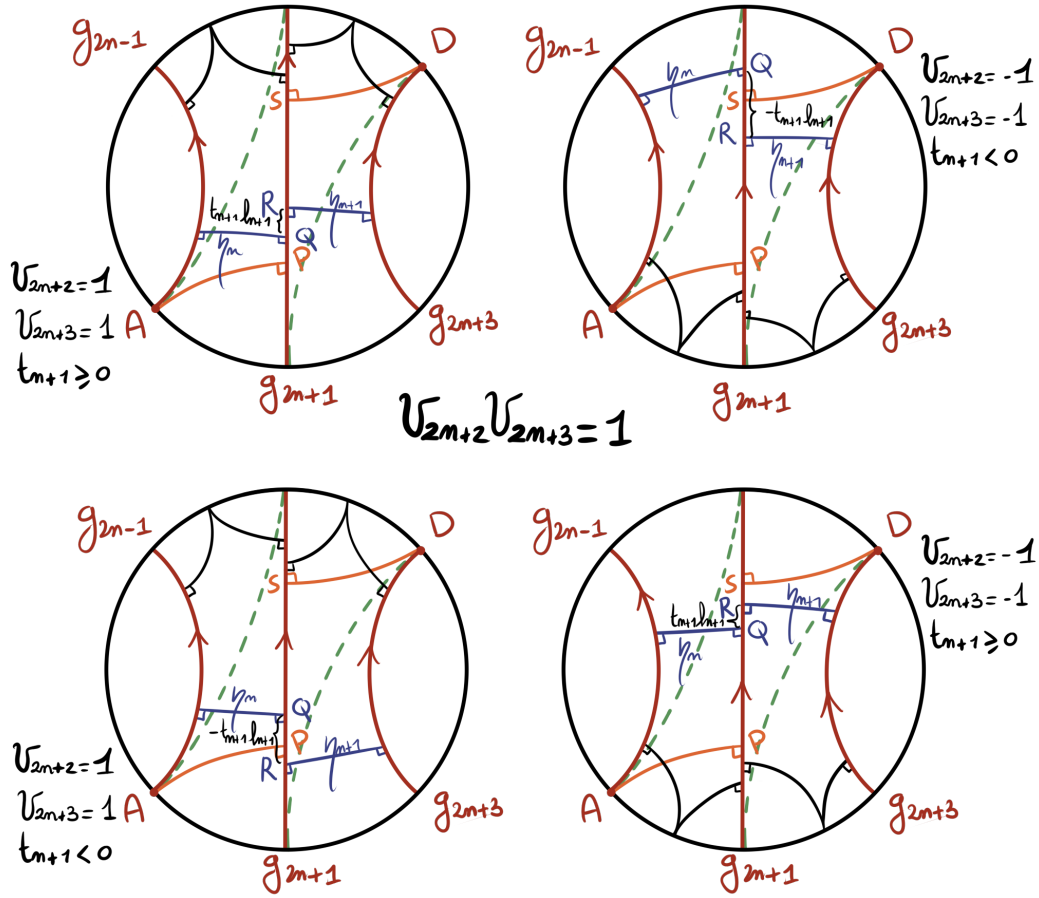


Figure 4.7: Configurations of pentagons around g_{2n+1} when $v_{2n+2} = v_{2n+3}$.

First, suppose $v_{2n+2}v_{2n+3} = 1$ (i.e. the pentagons have the same orientation), as in Figure 4.7. When $t_{n+1} < 0$, we have

$$\begin{aligned}
 s(g_{2n+1}) = \ell(PS) &= \ell(PQ) + \ell(RS) - \ell(QR) \\
 &= \ell(PQ) + \ell(RS) - (-t_{n+1}\ell_{n+1}) \\
 &= \ell(PQ) + \ell(RS) + t_{n+1}\ell_{n+1}.
 \end{aligned}$$

When $t_{n+1} \geq 0$, we have

$$\begin{aligned}
 s(g_{2n+1}) = \ell(PS) &= \ell(PQ) + \ell(RS) + \ell(QR) \\
 &= \ell(PQ) + \ell(RS) + t_{n+1}\ell_{n+1}.
 \end{aligned}$$

Thus for $v_{2n+2}v_{2n+3} = 1$, we have

$$s(g_{2n+1}) = \sinh^{-1} \left(\frac{1}{\sinh \ell(\eta_n)} \right) + \sinh^{-1} \left(\frac{1}{\sinh \ell(\eta_{n+1})} \right) + u_{n+1}\ell_{n+1}.$$

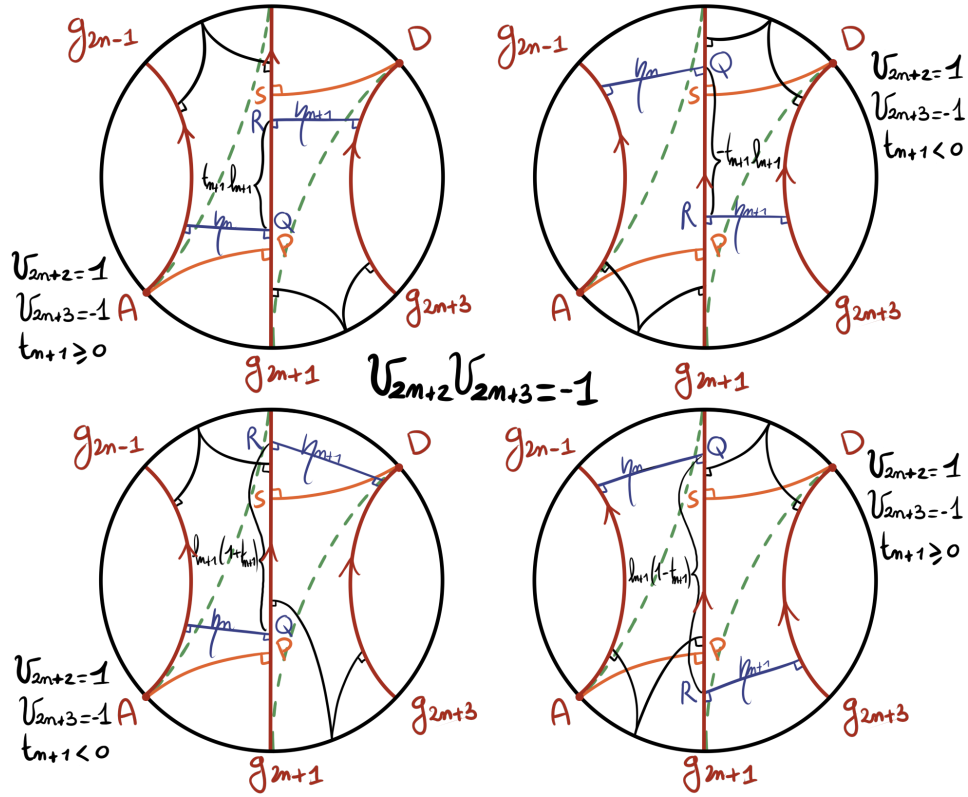


Figure 4.8: Configurations of pentagons around g_{2n+1} when $v_{2n+2} \neq v_{2n+3}$.

Then, suppose $v_{2n+2}v_{2n+3} = -1$ (i.e. the pentagons have different orientations), as in Figure 4.8. When $v_{2n+2}t_{n+1} > 0$ we have

$$\begin{aligned} s(g_{2n+1}) = \ell(PS) &= \ell(PQ) + \ell(RS) + v_{2n+2}\ell(QR) \\ &= \ell(PQ) + \ell(RS) + v_{2n+2}(v_{2n+2}t_{n+1})\ell_{n+1} \\ &= \ell(PQ) + \ell(RS) + t_{n+1}\ell_{n+1}. \end{aligned}$$

When $v_{2n+2}t_{n+1} < 0$, we have

$$\begin{aligned} s(g_{2n+1}) = \ell(PS) &= \ell(PQ) + \ell(RS) + v_{2n+2}\ell(QR) \\ &= \ell(PQ) + \ell(RS) + v_{2n+2}(1 + v_{2n+2}t_{n+1})\ell_{n+1} \\ &= \ell(PQ) + \ell(RS) + (v_{2n+2} + t_{n+1})\ell_{n+1}. \end{aligned}$$

Thus for $v_{2n+2}v_{2n+3} = -1$, we have

$$s_{2n+1} = s(g_{2n+1}) = \sinh^{-1} \left(\frac{1}{\sinh \ell(\eta_n)} \right) + \sinh^{-1} \left(\frac{1}{\sinh \ell(\eta_{n+1})} \right) + u_{n+1}\ell_{n+1}.$$

The other configurations with $w_{2n+1} + w_{2n+2} + w_{2n+3} + w_{2n+4} = 0$ correspond to those where the orthoray crosses the orthogonal between the boundaries of P_{n+1} or P_{n+2} . Then the computation of the shear does not differ to what we just did, as long as we take well into consideration the fact that $\tilde{\sigma}$ goes from one pentagon to the other one inside one pair of pants.

However, when $w_{2n+1} + w_{2n+2} + w_{2n+3} + w_{2n+4} \neq 0$, things are different and there are many different configurations to take into account, see Figure 4.9.

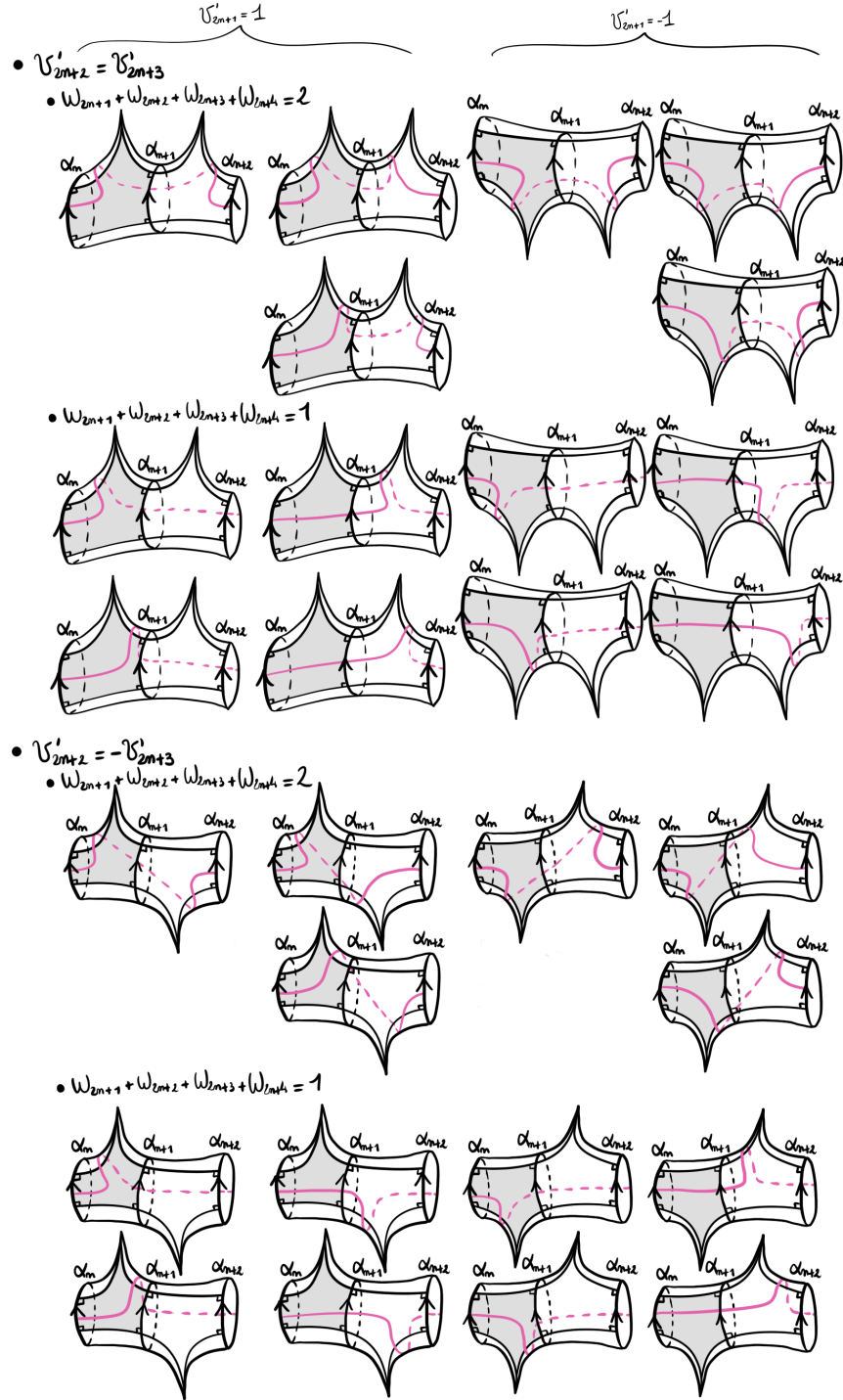


Figure 4.9: The 28 possible configurations for $w_{2n+1} + w_{2n+2} + w_{2n+3} + w_{2n+4} \neq 0$ and $w_{2n+2} + w_{2n+3} \neq 2$.

In Figure 4.9, all the twists are equal to 0 or $\frac{1}{2}$ to keep the figure simple. The reader should think about twists equal to 0 as corresponding to configurations where $v_{2n+2} = v_{2n+3}$, while twists equal to $\frac{1}{2}$ as corresponding to configurations where $v_{2n+2} = -v_{2n+3}$.

As before, for each configuration, the shear is the sum of $\ell(PQ)$ and $\ell(RS)$ plus or minus $\ell(QR)$. Let us compute the shear when $w_{2n+2} + w_{2n+3} \neq 2$.

Case $v_{2n+2} = v_{2n+3}$ and $w_{2n+2} + w_{2n+3} = 0$

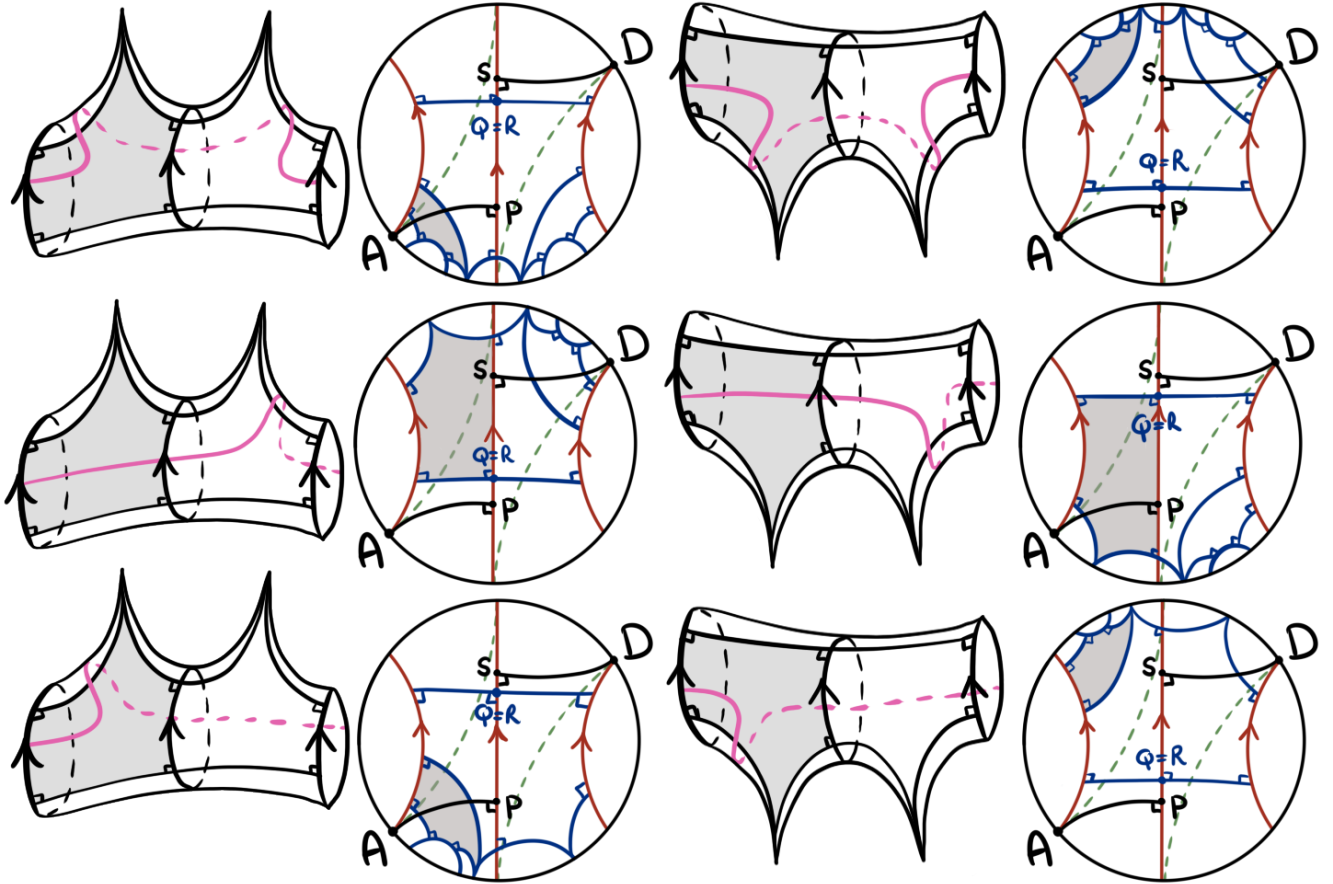


Figure 4.10: Configurations when $v_{2n+2} = v_{2n+3}$ and $w_{2n+2} + w_{2n+3} = 0$.

In these cases, $\ell(QR) = \ell_{n+1}|t_{n+1}|$. If $t_{n+1} \geq 0$, then $s_{2n+1} = \ell(PQ) + \ell(RS) + \ell(QR)$ otherwise $s_{2n+1} = \ell(PQ) + \ell(RS) - \ell(QR)$. Thus, we have

$$s_{2n+1} = \sinh^{-1} \left(\frac{1}{\sinh \ell(\eta_n)} \right) + \sinh^{-1} \left(\frac{1}{\sinh \ell(\eta_{n+1})} \right) + \ell_{n+1} u_{n+1}.$$

Case $v_{2n+2} = v_{2n+3}$ and $v_{2n+1}(1 - 2w_{2n+1}) = 1$

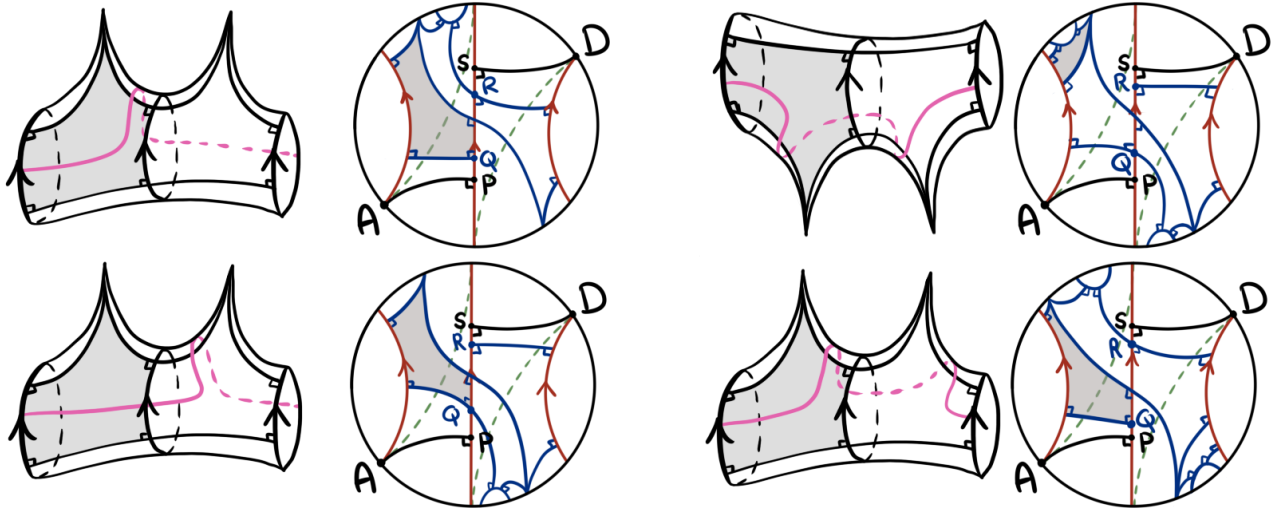


Figure 4.11: Configurations when $v_{2n+2} = v_{2n+3}$ and $w_{2n+2} + w_{2n+3} = 1$.

In these cases, $s_{2n+1} = \ell(PQ) + \ell(RS) + \ell(QR)$ and $\ell(QR) = \ell_{n+1}(1 + t_{n+1})$. Thus, we have

$$s_{2n+1} = \sinh^{-1} \left(\frac{1}{\sinh \ell(\eta_n)} \right) + \sinh^{-1} \left(\frac{1}{\sinh \ell(\eta_{n+1})} \right) + \ell_{n+1} u_{n+1}.$$

Case $v_{2n+2} = v_{2n+3}$ and $v_{2n+1}(1 - 2w_{2n+1}) = -1$

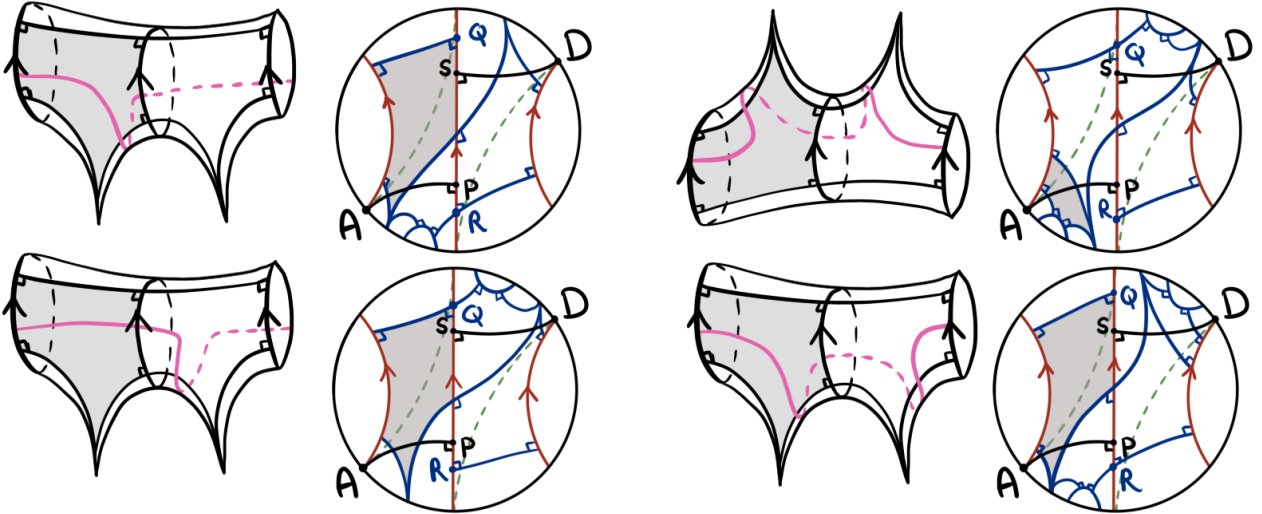


Figure 4.12: Configurations when $v_{2n+2} = v_{2n+3}$ and $v_{2n+1}(1 - 2w_{2n+1}) = -1$.

In these cases, $s_{2n+1} = \ell(PQ) + \ell(RS) - \ell(QR)$ and $\ell(QR) = \ell_{n+1}(1 - t_{n+1})$. Thus, we have

$$s_{2n+1} = \sinh^{-1} \left(\frac{1}{\sinh \ell(\eta_n)} \right) + \sinh^{-1} \left(\frac{1}{\sinh \ell(\eta_{n+1})} \right) + \ell_{n+1} u_{n+1}.$$

Case $v_{2n+2} \neq v_{2n+3}$ and $v_{2n+2}(1 - 2(w_{2n+2} + w_{2n+3})) > 0$

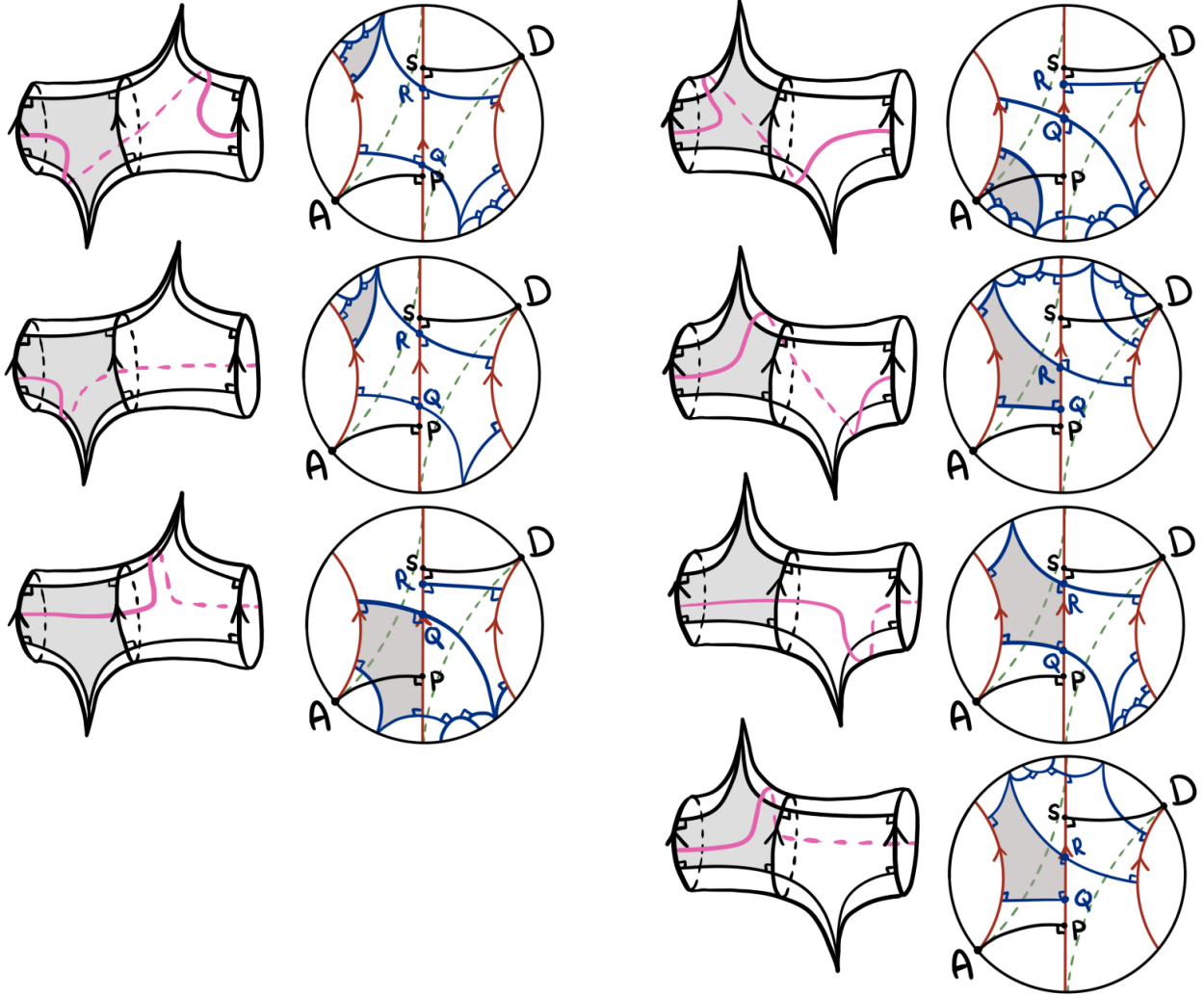


Figure 4.13: Configurations when $v_{2n+2} \neq v_{2n+3}$ and $v_{2n+2}(1 - 2(w_{2n+2} + w_{2n+3})) > 0$.

In these cases, $s_{2n+1} = \ell(PQ) + \ell(RS) + \ell(QR)$. When $t_{n+1} > 0$, we have $\ell(QR) = \ell_{n+1}t_{n+1}$ and when $t_{n+1} < 0$, we have $\ell(QR) = \ell_{n+1}(1 + t_{n+1})$. Thus, we have

$$s_{2n+1} = \sinh^{-1} \left(\frac{1}{\sinh \ell(\eta_n)} \right) + \sinh^{-1} \left(\frac{1}{\sinh \ell(\eta_{n+1})} \right) + \ell_{n+1}u_{n+1}.$$

Case $v_{2n+2} \neq v_{2n+3}$ and $v_{2n+2}(1 - 2(w_{2n+2} + w_{2n+3})) < 0$

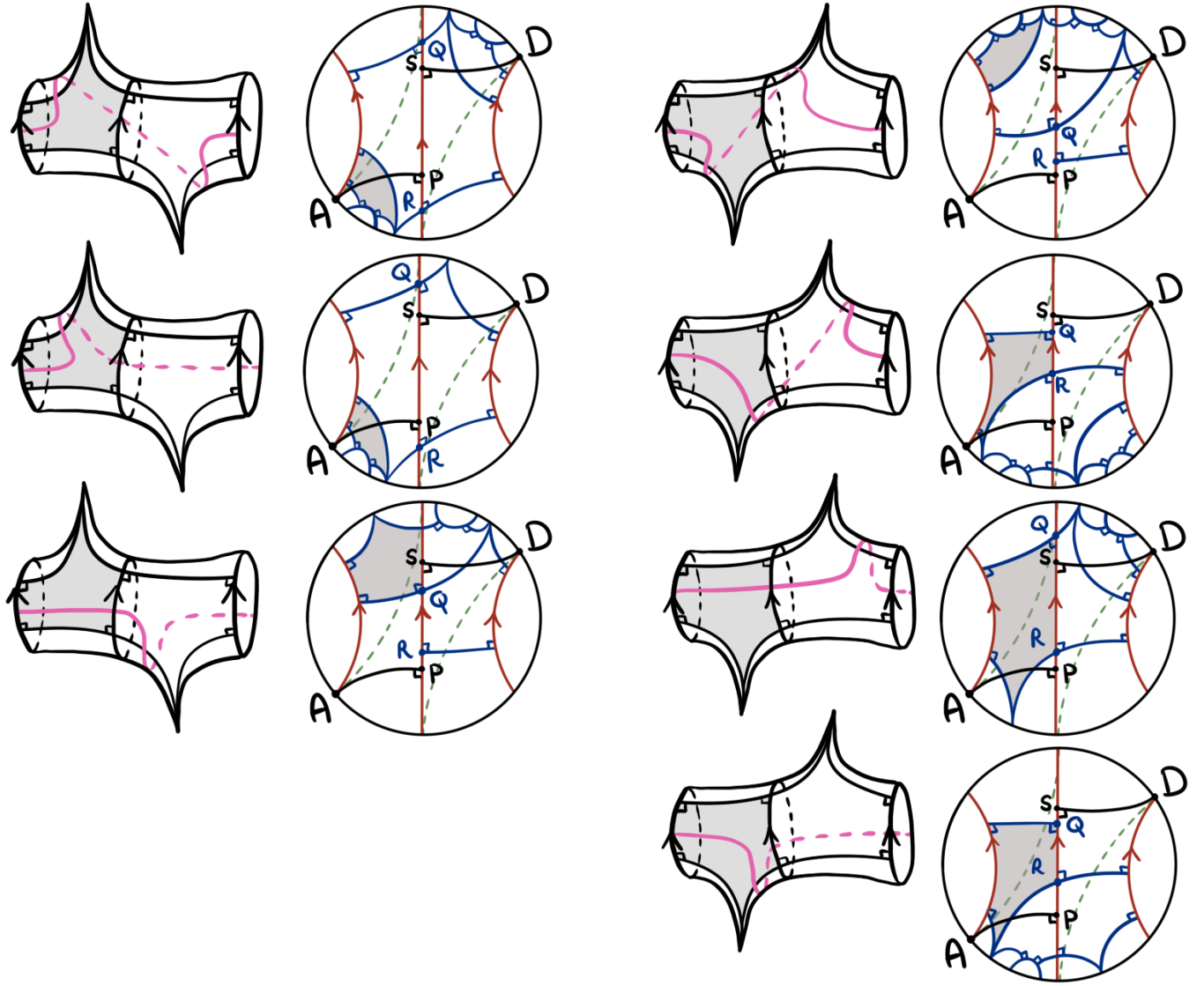


Figure 4.14: Configurations when $v_{2n+2} \neq v_{2n+3}$ and $v_{2n+2}(1 - 2(w_{2n+2} + w_{2n+3})) < 0$.

In these cases, $s_{2n+1} = \ell(PQ) + \ell(RS) - \ell(QR)$. When $t_{n+1} > 0$, we have $\ell(QR) = \ell_{n+1}(1 - t_{n+1})$ and when $t_{n+1} < 0$, we have $\ell(QR) = -\ell_{n+1}t_{n+1}$. Thus, we have

$$s_{2n+1} = \sinh^{-1} \left(\frac{1}{\sinh \ell(\eta_n)} \right) + \sinh^{-1} \left(\frac{1}{\sinh \ell(\eta_{n+1})} \right) + \ell_{n+1}u_{n+1}.$$

In conclusion, when $w_{2n+2} + w_{2n+3} \neq 2$, and for all of the previous configurations, we have

$$s_{2n+1} = \sinh^{-1} \left(\frac{1}{\sinh \ell(\eta_n)} \right) + \sinh^{-1} \left(\frac{1}{\sinh \ell(\eta_{n+1})} \right) + \ell_{n+1}u_{n+1}. \quad (4.3)$$

Step 2: Bounding the length of h .

We will use the following three inequalities which hold for x small enough:

$$x \leq \sinh(x) \leq x \sinh(1), \quad (4.4)$$

$$\frac{2}{x} \leq e^{\sinh^{-1}\left(\frac{1}{\sinh(x)}\right)} \leq \frac{5}{x} \quad (4.5)$$

and

$$e^{\sinh^{-1}\left(\frac{1}{\sinh(x)}\right)} \sinh\left(\frac{x}{2}\right) < 1 + x. \quad (4.6)$$

Now let us compute a lower bound on $\sum_{n=1}^{\infty} e^{s_{2n}+s_{2n-1}+\dots+s_1}$. The sum can be rewritten as $\sum_{n=1}^{\infty} \prod_{k=0}^{n-1} e^{s_{2(k+1)}+s_{2k+1}}$. With the computations of the shear (4.2) and (4.3), we have

$$\begin{aligned} \sum_{n=1}^{\infty} e^{s_{2n}+s_{2n-1}+\dots+s_1} &= \sum_{n=1}^{\infty} \prod_{k=0}^{n-1} e^{s_{2(k+1)}+s_{2k+1}} \\ &\geq C_1 \sum_{n=1}^{\infty} \prod_{k=1}^{n-1} \sinh\left(\frac{\ell(\eta_{k+1})}{2}\right)^2 e^{\sinh^{-1}\left(\frac{1}{\sinh(\ell(\eta_k))}\right)} e^{\sinh^{-1}\left(\frac{1}{\sinh(\ell(\eta_{k+1}))}\right)} e^{u_{k+1}\ell_{k+1}} \end{aligned}$$

By (4.4) we have $\frac{\ell(\eta_{k+1})^2}{4} \leq \sinh\left(\frac{\ell(\eta_{k+1})}{2}\right)^2$ and by (4.5) we obtain:

$$\begin{aligned} \sum_{n=1}^{\infty} e^{s_{2n}+s_{2n-1}+\dots+s_1} &\geq C_1 \sum_{n=1}^{\infty} \prod_{k=1}^{n-1} \frac{\ell(\eta_{k+1})^2}{4} \frac{2}{\ell(\eta_k)} \frac{2}{\ell(\eta_k+1)} e^{u_{k+1}\ell_{k+1}} \\ &\geq C_1 \sum_{n=1}^{\infty} \prod_{k=1}^{n-1} \frac{\ell(\eta_{k+1})}{\ell(\eta_k)} e^{u_{k+1}\ell_{k+1}} \\ &\geq C_2 \sum_{n=1}^{\infty} \ell(\eta_n) e^{u_n\ell_n+\dots+u_1\ell_1} \end{aligned}$$

Then, let us compute a lower bound on $\sum_{n=1}^{\infty} e^{-s_{2n+1}-s_{2n}-\dots-s_1}$. With the computations of the shear (4.2) and (4.3), we have

$$\begin{aligned} \sum_{n=1}^{\infty} e^{-s_{2n+1}-s_{2n}-\dots-s_1} &= \sum_{n=1}^{\infty} e^{-(u_{n+1}\ell_{n+1}+\dots+u_1\ell_1)} \left[e^{-\sinh^{-1}\left(\frac{1}{\ell(\eta_n)}\right)} e^{-\sinh^{-1}\left(\frac{1}{\ell(\eta_{n+1})}\right)} \right] \frac{1}{\sinh^2\left(\frac{\ell(\eta_n)}{2}\right)} \dots \\ &\quad \left[e^{-\sinh^{-1}\left(\frac{1}{\ell(\eta_1)}\right)} e^{-\sinh^{-1}\left(\frac{1}{\ell(\eta_2)}\right)} \right] \frac{1}{\sinh^2\left(\frac{\ell(\eta_1)}{2}\right)} e^{-s_1}. \end{aligned}$$

By inequality (4.6), we have

$$\begin{aligned} \sum_{n=1}^{\infty} e^{-s_{2n+1}-s_{2n}-\dots-s_1} &\geq C_3 \sum_{n=1}^{\infty} e^{-(u_{n+1}\ell_{n+1}+\dots+u_1\ell_1)} \frac{e^{-\sinh(\frac{1}{\sinh(\ell(\eta_{n+1})))}}{\sinh\left(\frac{\ell(\eta_1)}{2}\right)} \left(\prod_{k=1}^n \frac{1}{1+\ell(\eta_k)}\right)^2 \\ &\geq C_4 \sum_{n=1}^{\infty} e^{-(u_{n+1}\ell_{n+1}+\dots+u_1\ell_1)} e^{-\sinh(\frac{1}{\sinh(\ell(\eta_{n+1})))}} \left(\prod_{k=1}^n \frac{1}{1+\ell(\eta_k)}\right)^2. \end{aligned}$$

Then by inequalities (4.5) and (4.1), we obtain

$$\begin{aligned} \sum_{n=1}^{\infty} e^{-s_{2n+1}-s_{2n}-\dots-s_1} &\geq C_4 \sum_{n=1}^{\infty} e^{-(u_{n+1}\ell_{n+1}+\dots+u_1\ell_1)} \frac{\ell(\eta_{n+1})}{5} \left(\prod_{k=1}^n \frac{1}{1+\ell(\eta_k)}\right)^2 \\ &\geq C_5 \sum_{n=1}^{\infty} \ell(\eta_{n+1}) e^{-(u_{n+1}\ell_{n+1}+\dots+u_1\ell_1)} \end{aligned}$$

By combining the two sums, we obtain the following lower bound on $\ell(h)$:

$$\begin{aligned} \ell(h) &\geq C \left(\sum_{n=1}^{\infty} \ell(\eta_n) e^{u_n\ell_n+\dots+u_1\ell_1} + \sum_{n=1}^{\infty} \ell(\eta_n) e^{-(u_n\ell_n+\dots+u_1\ell_1)} \right) \\ &\geq C \sum_{n=1}^{\infty} \ell(\eta_n) \cosh(u_n\ell_n + \dots + u_1\ell_1) \end{aligned}$$

Let us continue by computing an upper bound on $\sum_{n=1}^{\infty} e^{s_{2n+1}+s_{2n}+\dots+s_1}$. We have

$$\begin{aligned} \sum_{n=1}^{\infty} e^{s_{2n+1}+s_{2n}+\dots+s_1} &= \sum_{n=1}^{\infty} e^{u_n\ell_n+\dots+u_1\ell_1} \left[e^{\sinh^{-1}\left(\frac{1}{\ell(\eta_n)}\right)} e^{\sinh^{-1}\left(\frac{1}{\ell(\eta_{n-1})}\right)} \right] \sinh^2\left(\frac{\ell(\eta_n)}{2}\right) \dots \\ &\quad \left[e^{\sinh^{-1}\left(\frac{1}{\ell(\eta_1)}\right)} e^{\sinh^{-1}\left(\frac{1}{\ell(\eta_2)}\right)} \right] \sinh^2\left(\frac{\ell(\eta_1)}{2}\right) e^{s_1}. \end{aligned}$$

By inequalities (4.6) and (4.1), we obtain

$$\begin{aligned} \sum_{n=1}^{\infty} e^{s_{2n+1}+s_{2n}+\dots+s_1} &\leq C'_1 \sum_{n=1}^{\infty} e^{u_n\ell_n+\dots+u_1\ell_1} \sinh\left(\frac{\ell(\eta_n)}{2}\right) \sinh\left(\frac{\ell(\eta_1)}{2}\right) \left(\prod_{k=1}^n 1+\ell(\eta_k)\right)^2 \\ &\leq C'_2 \sum_{n=1}^{\infty} e^{u_n\ell_n+\dots+u_1\ell_1} \sinh\left(\frac{\ell(\eta_n)}{2}\right) \left(\prod_{k=1}^n 1+\ell(\eta_k)\right)^2 \\ &\leq C'_3 \sum_{n=1}^{\infty} e^{u_n\ell_n+\dots+u_1\ell_1} \sinh\left(\frac{\ell(\eta_n)}{2}\right). \end{aligned}$$

By (4.4), we have $\sinh\left(\frac{\ell(\eta_n)}{2}\right) \leq \frac{\ell(\eta_n)}{2} \sinh(1)$, so we have the following upper bound:

$$\sum_{n=1}^{\infty} e^{s_{2n+1}+s_{2n}+\dots+s_1} \leq C'_4 \sum_{n=1}^{\infty} \ell(\eta_n) e^{u_n\ell_n+\dots+u_1\ell_1}.$$

Finally, let us compute an upper bound for $\sum_{n=1}^{\infty} e^{-s_{2n+1}-s_{2n}-\dots-s_1}$. We write the sum as

$$\sum_{n=1}^{\infty} \prod_{k=1}^n e^{-s_{2k+1}-s_{2k}}.$$

With the computations of the shear, we have

$$\begin{aligned} \sum_{n=1}^{\infty} e^{-s_{2n+1}-s_{2n}-\dots-s_1} &\leq C'_5 \sum_{n=1}^{\infty} \prod_{k=1}^n e^{-s_{2k+1}-s_{2k}} \\ &\leq C'_5 \sum_{n=1}^{\infty} \prod_{k=0}^n \frac{1}{\sinh\left(\frac{\ell(\eta_k)}{2}\right)} e^{2 - \sinh^{-1}\left(\frac{1}{\sinh(\ell(\eta_k))}\right) - \sinh^{-1}\left(\frac{1}{\sinh(\ell(\eta_{k+1}))}\right)} e^{-u_{k+1}\ell_{k+1}}. \end{aligned}$$

By (4.4), we have $\frac{1}{\sinh(\frac{\ell(\eta_k)}{2})} \leq \frac{2}{\ell(\eta_k)}$, and by (4.5) we obtain

$$\begin{aligned} \sum_{n=1}^{\infty} e^{-s_{2n+1}-s_{2n}-\dots-s_1} &\leq C'_5 \sum_{n=1}^{\infty} \prod_{k=1}^n \frac{4}{\ell(\eta_k)^2} \frac{\ell(\eta_{k+1})}{2} \frac{\sinh(\ell(\eta_k))}{2} e^{-u_{k+1}\ell_{k+1}} \\ &\leq C'_5 \sum_{n=1}^{\infty} \prod_{k=1}^n \frac{\ell(\eta_{k+1})}{\ell(\eta_k)} e^{-u_{k+1}\ell_{k+1}} \\ &\leq C'_6 \sum_{n=1}^{\infty} \ell(\eta_{n+1}) e^{-(u_{n+1}\ell_{n+1}+\dots+u_1\ell_1)} \end{aligned}$$

By combining the two sums, we obtain the following upper bound on $\ell(h)$:

$$\begin{aligned} \ell(h) &\leq C' \left(\sum_{n=1}^{\infty} \ell(\eta_n) e^{u_n\ell_n+\dots+u_1\ell_1} + \sum_{n=1}^{\infty} \ell(\eta_n) e^{-(u_n\ell_n+\dots+u_1\ell_1)} \right) \\ &\leq C' \sum_{n=1}^{\infty} \ell(\eta_n) \cosh(u_n\ell_n + \dots + u_1\ell_1) \end{aligned}$$

In conclusion, the length of h is infinite if and only if the sum $\sum_{n=1}^{\infty} \ell(\eta_n) \cosh(u_n\ell_n + \dots + u_1\ell_1)$ is infinite.

Consider the pentagon between g_{2n-1} and g_{2n+1} : its sides contained in g_{2n-1} and g_{2n+1} are of lengths $\frac{\ell_n}{2}$ and $\frac{\ell_{n+1}}{2}$. Trace the orthogonal to η_n going through the vertex of the pentagon; note that it cuts the pentagon into two trirectangles. By (2.3), we obtain the following identity:

$$\ell(\eta_n) = \sinh^{-1}\left(\frac{1}{\sinh(\frac{\ell_n}{2})}\right) + \sinh^{-1}\left(\frac{1}{\sinh(\frac{\ell_{n+1}}{2})}\right).$$

By assumption, for n large enough, $\ell(\eta_n)$ is arbitrarily small. This implies that ℓ_n and ℓ_{n+1} are both large. Then we have $\ell(\eta_n) \asymp e^{-\frac{\ell_{n+1}}{2}} + e^{-\frac{\ell_n}{2}}$.

Thus $\ell(h) \asymp \sum_{n=1}^{\infty} \left(e^{-\frac{\ell_{n+1}}{2}} + e^{-\frac{\ell_n}{2}} \right) \cosh(u_n\ell_n + \dots + u_1\ell_1)$. □

From Lemma 4.1.4 and Proposition 4.1.5, we can deduce that if any sum listed in Theorem 4.2.2 is finite, then the flute surface is not of the first kind. More is true: we can also use Theorem 4.2.2 to characterize when the flute is of the first kind.

Theorem 4.2.3. *Let $X = \{\ell_n, t_n\}_{n \in \mathbb{N}^*}$ be a tight flute surface. There exists an infinite family $\mathcal{U}$ of explicit sequences, that only depends on the twists, such that X is of the first kind if and only if for all sequences $(u_n)_{n \in \mathbb{N}^*} \in \mathcal{U}$, we have*

$$\sum_{n=1}^{\infty} \left(e^{-\frac{\ell_{n+1}}{2}} + e^{-\frac{\ell_n}{2}} \right) \cosh(u_n \ell_n + \dots + u_1 \ell_1) = \infty$$

Proof. By Proposition 4.1.5, the surface X is of the first kind if and only if none of its ends are visible. Each end of X that is a cusp is not a visible end, thus X is of the first kind if and only if its single infinite-type end is not visible. By Lemma 4.1.4, this end is not visible if no orthoray based at α_1 that exits this end can leave $C(X)$, in other words, if all orthorays based at α_1 have infinite length in $C(X)$. Thus we do not have to take into account the longer orthorays.

By construction, each of the orthorays we are interested in is described by a patchwork $((v_n)_{n \in \mathbb{N}^*}, (w_n)_{n \in \mathbb{N}^*})$. By Theorem 4.2.2, orthorays associated to a patchwork with $w_{2n+2} + w_{2n+3} \neq 2$ for all n have infinite length in $C(X)$ if and only if

$$\sum_{n=1}^{\infty} \left(e^{-\frac{\ell_{n+1}}{2}} + e^{-\frac{\ell_n}{2}} \right) \cosh(u_n \ell_n + \dots + u_1 \ell_1) = \infty.$$

with $(u_n)_{n \in \mathbb{N}^*}$ the sequence associated to the patchwork as described in Theorem 4.2.2.

Let us show that orthorays with patchwork satisfying $w_{2n+2} + w_{2n+3} = 2$ for the same n result in sums equal or bigger than sums obtained with patchwork such that $w_{2n+2} + w_{2n+3} \neq 2$ for all n . We start by computing shears for configurations such that $w_{2n+2} + w_{2n+3} = 2$ and then compare them to the computation of shears in configuration with $w_{2n+2} + w_{2n+3} \neq 2$. See in Figure 4.15 an illustration of this comparison.

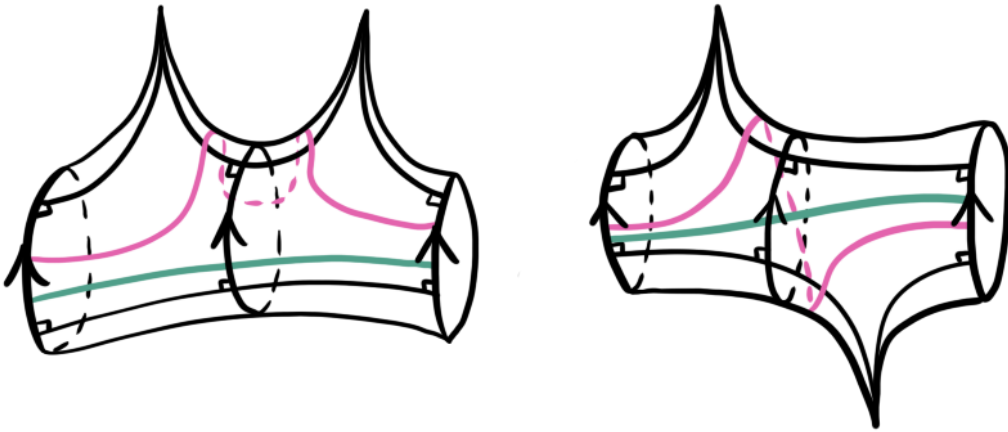


Figure 4.15: Comparison between configurations such that $w_{2n+2} + w_{2n+3} = 2$ and configurations such that $w_{2n+2} + w_{2n+3} \neq 2$.

- Configurations $v_{2n+1} = v_{2n+4} \neq v_{2n+2} = v_{2n+3}$ and $w_{2n+2} + w_{2n+3} = 2$. We can compare these configurations to configurations $v_{2n+1} = v_{2n+2} = v_{2n+3} = v_{2n+4}$ and $w_{2n+2} + w_{2n+3} = 0$. Both types of configurations lead to the same computation of the shears s_{2n-1} , s_{2n+1} and s_{2n+2} . Thus both configurations lead to the same sum.

Let us compute the shear s_{2n+1} .

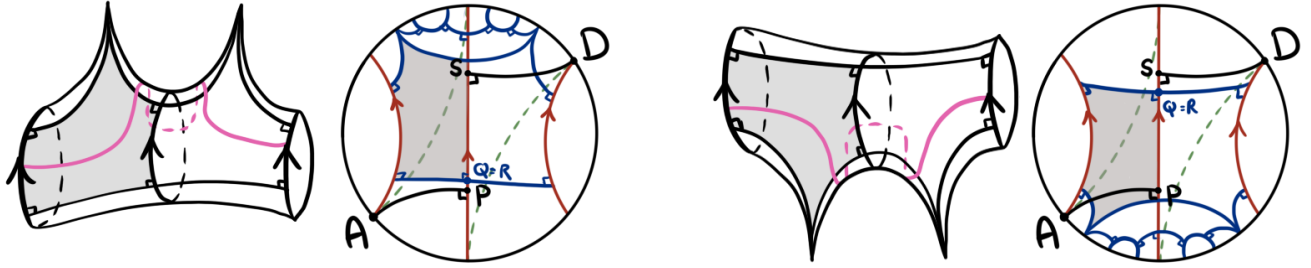


Figure 4.16: Configurations when $w_{2n+2} + w_{2n+3} = 2$ and $v_{2n+2} = v_{2n+3}$.

In these cases, $\ell(QR) = \ell_{n+1}|t_{n+1}|$. If $t_{n+1} \geq 0$ then $s_{2n+1} = \ell(PQ) + \ell(RS) + \ell(QR)$ otherwise $s_{2n+1} = \ell(PQ) + \ell(RS) - \ell(QR)$, which is the same result as in configurations described in Figure 4.7.

- Configurations $v_{2n+1} = v_{2n+3} \neq v_{2n+2} = v_{2n+4}$ and $w_{2n+2} + w_{2n+3} = 2$. We can compare these configurations to configurations $v_{2n+1} = v_{2n+2} = v_{2n+4} \neq v_{2n+3}$ and $w_{2n+2} + w_{2n+3} = 1$. Both types of configurations lead to the same computation of the shears s_{2n-1} , s_{2n+2} and s_{2n+1} . Thus both configurations lead to the same sum. In the second configuration, the orthoray goes from one pentagon to the other more than once in the second pair of pants, thus the orientation-reversing isometry in this pair of pants maps the orthoray to a piecewise geodesic path contained in a single pentagon. By pulling this piecewise geodesic tight we get a shorter orthoray, see Figure 4.17.

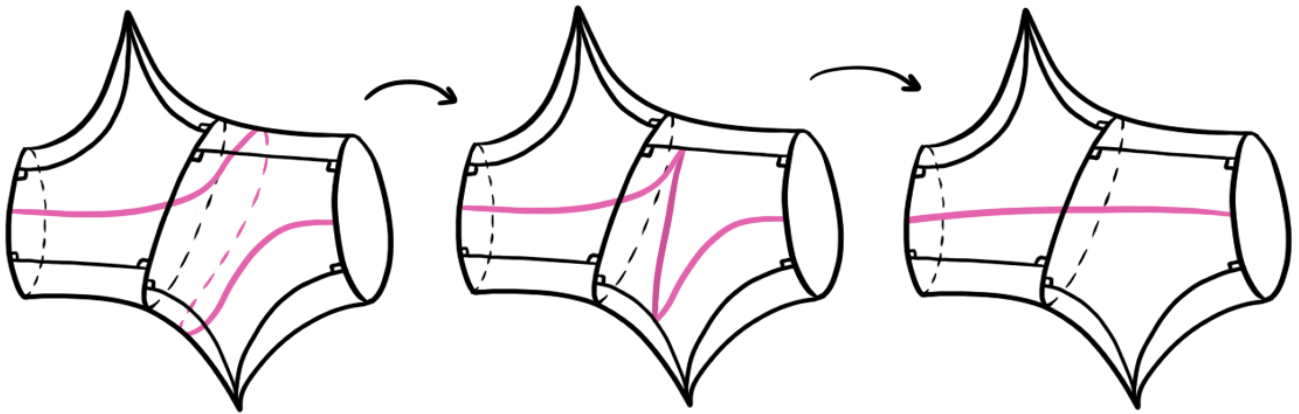


Figure 4.17: Construction of an orthoray shorter than in configurations $v_{2n+1} = v_{2n+2} = v_{2n+4} \neq v_{2n+3}$ and $w_{2n+2} + w_{2n+3} = 1$.

Let us compute the shear s_{2n+1} for $v_{2n+1} = v_{2n+3} \neq v_{2n+2} = v_{2n+4}$ and $w_{2n+2} + w_{2n+3} = 2$.

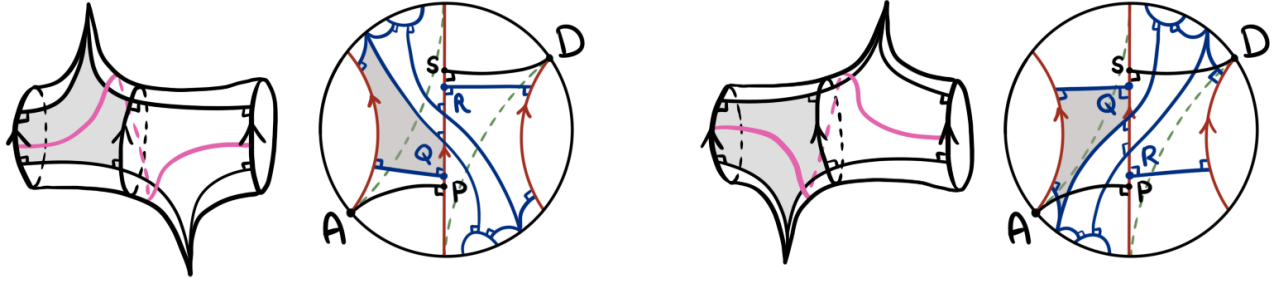


Figure 4.18: Configurations when $w_{2n+2} + w_{2n+3} = 2$ and $v_{2n+2} \neq v_{2n+3}$.

When $v_{2n+1} = 1$, we have $s_{2n+1} = \ell(PQ) + \ell(RS) + \ell(QR)$ and $\ell(QR) = \ell_{n+1}(1 + t_{n+1})$ if $t_{n+1} > 0$, $\ell(QR) = \ell_{n+1}(2 + t_{n+1})$ otherwise.

When $v_{2n+1} = -1$, we have $s_{2n+1} = \ell(PQ) + \ell(RS) - \ell(QR)$ and $\ell(QR) = \ell_{n+1}(2 - t_{n+1})$ if $t_{n+1} > 0$, $\ell(QR) = \ell_{n+1}(1 - t_{n+1})$ otherwise.

Let us compute the shear s_{2n+1} for $v'_{2n+1} = v_{2n+2} = v_{2n+4} \neq v_{2n+3}$ and $w_{2n+2} + w_{2n+3} = 1$.

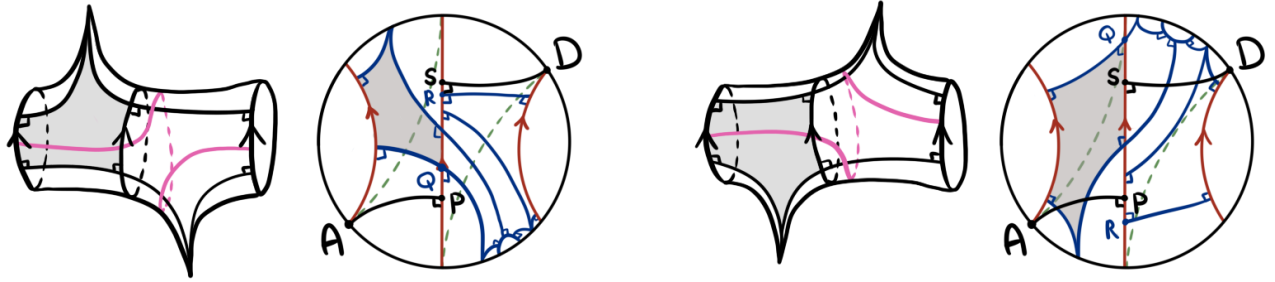


Figure 4.19: Configurations when $v_{2n+1} = v_{2n+2} = v_{2n+4} \neq v_{2n+3}$ and $w_{2n+2} + w_{2n+3} = 1$.

When $v_{2n+1} = 1$, we have $s_{2n+1} = \ell(PQ) + \ell(RS) + \ell(QR)$ and $\ell(QR) = \ell_{n+1}(1 + t_{n+1})$ if $t_{n+1} > 0$, $\ell(QR) = \ell_{n+1}(2 + t_{n+1})$ otherwise.

When $v_{2n+1} = -1$, we have $s_{2n+1} = \ell(PQ) + \ell(RS) - \ell(QR)$ and $\ell(QR) = \ell_{n+1}(2 - t_{n+1})$ if $t_{n+1} > 0$, $\ell(QR) = \ell_{n+1}(1 - t_{n+1})$ otherwise.

In conclusion, these four configurations can be ignored and only patchworks with $w_{2n+2} + w_{2n+3} \neq 2$ for all n have to be taken into account. $\square$

Under some conditions on the twist it is possible to reduce $\mathcal{U}$ to a single sequence in Theorem 4.2.3.

Corollary 4.2.4. *Let $X = \{\ell_n, t_n\}_{n \in \mathbb{N}^*}$ be a flute surface such that for all n we have $t_n \in \{0, \frac{1}{2}\}$. Then there exists a sequence $(u_n)_{n \in \mathbb{N}}$ such that X is of the first kind if and only if*

$$\sum_{n=1}^{\infty} \left(e^{-\frac{\ell_{n+1}}{2}} + e^{-\frac{\ell_n}{2}} \right) \cosh(u_n \ell_n + \dots + u_1 \ell_1) = \infty.$$

Proof. When all the twists are 0 or $\frac{1}{2}$, the pentagons of each pair of pants are aligned with each other as in Figure 4.20.

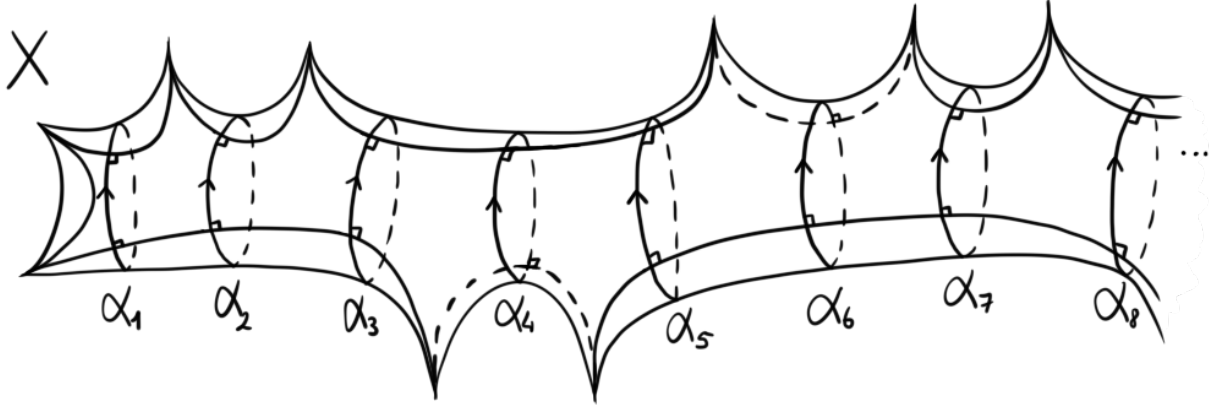


Figure 4.20: Flute surface whose twists parameters are in $\{0, 1/2\}$.

Choose either P_2^1 or P_2^{-1} as the starting pentagon $P_2^{v_{2n}}$. We then define the front side patchwork as follows: for all $n > 0$, $v_{2n+2} = v_{2n+1}$, then we have $v_{2n+1} = v_{2n}$ if $t_n = 0$, and we have $v_{2n+1} = -v_{2n}$ if $t_n = \frac{1}{2}$. This implies that $w_n = 0$ for all n . We call $X^* = \cup_{n>1} P_n^{v_{2n-1}}$ the front side. The union of all of the other pentagons that are not in the front side patchwork is called the back side.

There is an orientation-reversing isometry R between the front side X^* and the back side that fixes the points in the intersection between the two sides (in [PŠ23] the authors call a surface with such an isometry *symmetric*). Thus, we can map any geodesic of X into a piecewise geodesic path of the same length in X^* : we keep the part of the geodesic that is already in the front side and we use the isometry R to map the part of the geodesic that is outside of the front side, inside X^* .

Thus any orthoray on X based at α_1 can be mapped onto a piecewise geodesic in the homotopy class relative to α_1 of the orthoray associated to the front side patchwork. This implies that there are two isometric orthorays, the ones associated to the front patchwork side and to the back side patchwork, that are shorter than any other orthorays. So by Theorem 4.2.3, it suffices to check if the sum

$$\sum_{n=1}^{\infty} \left(e^{-\frac{\ell_{n+1}}{2}} + e^{-\frac{\ell_n}{2}} \right) \cosh(u_n \ell_n + \dots + u_1 \ell_1)$$

associated to the front side patchwork goes to infinity to know if X is of the first kind.

□

Note that in the case of the twist being 0 or $\frac{1}{2}$, it is possible to really simplify the sequence $(u_n)_{n \in \mathbb{N}^*}$:

$$u_n = \begin{cases} 0 & \text{if } t_n = 0, \\ v_{2n} + \frac{1}{2} & \text{if } t_n = \frac{1}{2} \end{cases}$$

for all n .

In the case of non-tight flute surfaces, the presence of funnels ensures that the surface is not of the first kind. However we can still derive from the proofs of Theorem 4.2.2 and Theorem 4.2.3 a way to check if the infinite-type end of the flute is visible or not. The only change (aside from the impossibility to apply Proposition 4.1.5) is the computation of $\ell(\eta_n)$ in the proof of Theorem 4.2.2. If the isolated end between α_{n-1} and α_n is a funnel, denote by γ_n the closed geodesic bounding the funnel and set $c_n = \ell_X(\gamma_n)$ (where $c_n = 0$ if the corresponding end is a cusp). The pair of pants P_{n+1} is now cut into two right-angled hexagons (instead of pentagons). We describe this new flute as $X = \{\ell_n, t_n, c_n\}_{n \in \mathbb{N}^*}$. With Lemma 2.3.7 we compute the length of the orthogeodesic in P_{n+1} between α_n and α_{n+1} when $c_n \neq 0$ and we define the sequence $(L_n)_{n \in \mathbb{N}^*}$ as follows:

$$L_n = \begin{cases} e^{-\frac{\ell_{n+1}}{2}} + e^{-\frac{\ell_n}{2}} & \text{if } c_n = 0, \\ \cosh^{-1} \left(\frac{\cosh(\frac{c_n}{2}) + \cosh(\frac{\ell_n}{2}) \cosh(\frac{\ell_{n+1}}{2})}{\sinh(\frac{\ell_n}{2}) \sinh(\frac{\ell_{n+1}}{2})} \right) & \text{otherwise.} \end{cases}$$

And we obtain the following corollary:

Corollary 4.2.5. *Let $X = \{\ell_n, t_n, c_n\}_{n \in \mathbb{N}^*}$ be a flute surface. There exists an infinite family $\mathcal{U}$ of explicit sequences, only depending on twist parameters, such that the infinite-type end of X is not visible if and only if for all sequences $(u_n)_{n \in \mathbb{N}^*} \in \mathcal{U}$, we have*

$$\sum_{n=1}^{\infty} L_n \cosh(u_n \ell_n + \dots + u_1 \ell_1) = \infty,$$

where L_n is as above.

Theorem 4.2.3 can also be used to deduce if more complicated surfaces have visible or non-visible ends and if they are of the first kind.

Corollary 4.2.6. *Let $N \in \mathbb{N}$.*

Let X be an infinite-type hyperbolic surface with only planar ends such that isolated ends are cusps and denote $\{e_i\}_{i=1}^N$ its non-isolated ends. Write X as a union $X = \Sigma \cup \bigcup_{i=1}^N X_i$ where Σ is a N -holed sphere and each X_i is a tight flute surface with one boundary component α_1^i with e_i as its infinite-type end. Let $\{\ell_n^i, t_n^i\}_{n \in \mathbb{N}^}$ be the Fenchel-Nielsen coordinates of X_i given by a pants decomposition that cuts X_i into a collection of pair of pants, with one cusp and two boundary components α_k^i and α_{k+1}^i with $k > 0$.*

Then there exist N infinite families $\mathcal{U}_i$ of explicit sequences such that X is of the first kind if and only if, for all $1 \leq i \leq N$ and for all sequences $(u_n^i)_{n \in \mathbb{N}^} \in \mathcal{U}_i$ we have*

$$\sum_{n=1}^{\infty} \left(e^{-\frac{\ell_{n+1}^i}{2}} + e^{-\frac{\ell_n^i}{2}} \right) \cosh(u_n^i \ell_n^i + \dots + u_1^i \ell_1^i) = \infty.$$

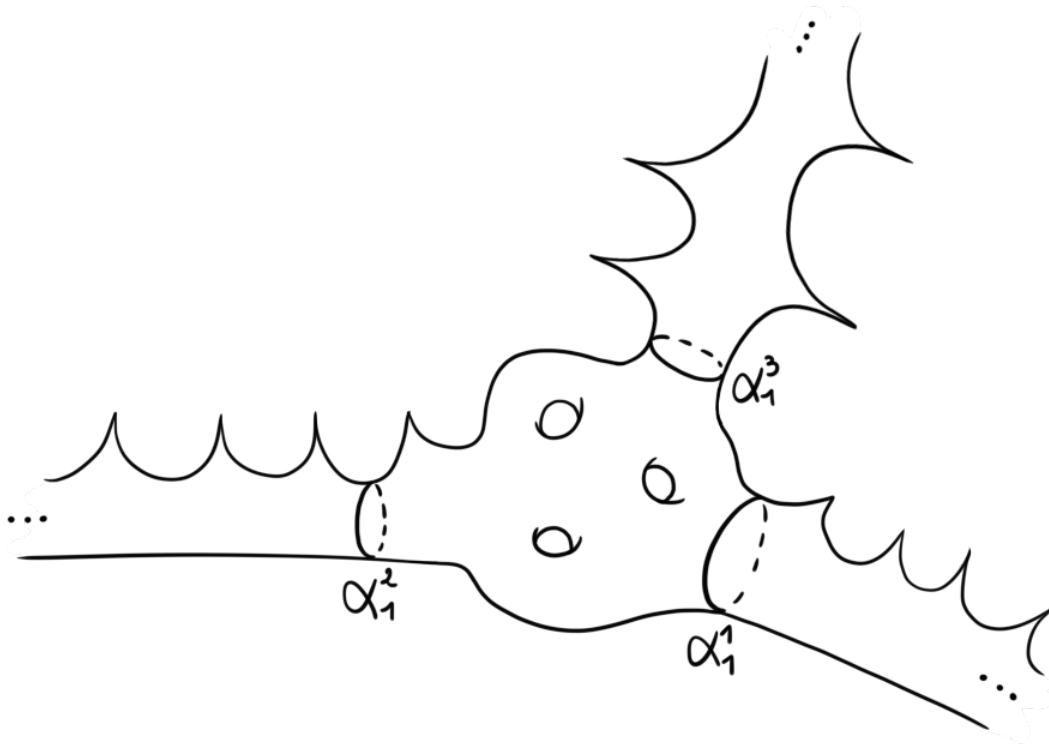


Figure 4.21: Example of surface on which we can apply Corollary 4.2.6.

To prove this corollary it suffices to check if each end is visible or not in the same way as we did for the tight flute surface. For each end e_i , we base orthorays that exit e_i at α_1^i and the rest of the proof follows.

4.3 Application: parabolic tight flute surfaces

As described in [AS60, Chap 4.6], by the uniformization Theorem 1, simply connected surfaces can be classified into elliptic, parabolic and hyperbolic surfaces. Parabolic surfaces are conformally equivalent to the complex plane, and unlike simply connected hyperbolic surfaces, they do not admit a Green's function [AS60].

Definition 4.3.1. Let X be a Riemannian surface. A *Green's function* for X with pole at a point $p \in X$ is a function

$$G_p: S \setminus \{p\} \rightarrow \mathbb{R}$$

that is harmonic on $X \setminus \{p\}$, vanishes at the boundary of X , and has a logarithmic singularity at p , that is,

$$G_p(x) \sim \log |x - p| \quad \text{as } x \rightarrow p.$$

We say that X *admits a Green's function* if such a function exists for some point $p \in X$.

Thus this property can be used to extend the definition of parabolicity to arbitrary open surfaces.

Definition 4.3.2. A hyperbolic surface X is said to be *parabolic* if it does not admit a Green's function.

Note that this extension of the definition of parabolicity is independent from the classification of surfaces by their universal cover as in Theorem 2. Thus, a parabolic hyperbolic surface in our context is not a contradiction. To avoid any ambiguity, surfaces that admit a Green's function will simply be called "non-parabolic" surfaces.

This notion of parabolicity is very interesting, as it has many equivalent characterizations reflecting different mathematical perspectives.

Theorem 4.3.3. *Let X be a Riemann surface, X being of parabolic type is equivalent to :*

- *the boundary at infinity has zero harmonic measure [AS60],*
- *the Poincaré series $\sum_{\gamma \in \Gamma} e^{-d(0, \gamma(0))}$ is divergent, where $X = \mathbb{H}/\Gamma$, $d(.,.)$ is the hyperbolic distance in $\mathbb{H}$ [Tsu75, Gar86],*
- *the Brownian motion on X is recurrent [FM01],*
- *the geodesic flow on the unit tangent bundle $T^1(X)$ of X is ergodic [Nic89],*
- *the limit set of a quasiconformal deformation of Γ has Bowen's property [Bis01, AZ90], and*
- *almost every horizontal trajectory of every finite area holomorphic quadratic differential on X is recurrent [vS24].*

Moreover, in the case of tight flutes surfaces, the notion of parabolicity still has a strong link with its root definition. Indeed, a parabolic tight flute surface is conformal to the complex plane with countably infinitely many isolated points removed and a non-parabolic tight flute surface is conformal to the interior of a disk with countably infinitely many isolated points removed [MR17].

As the complex plane with countably infinitely many isolated points removed and the interior of a disk with countably infinitely many isolated points removed are not quasiconformal, we also remark that the Deformation space of tight flute surfaces equipped with the metric d_{Def} defined in Section 2.2, is not connected.

The link between first kind surfaces and parabolic surfaces is the following:

Proposition 4.3.4. *If a surface X is parabolic, then it is of the first kind.*

There are several ways to show this proposition, depending on which equivalent characterization from Theorem 4.3.3 we want to use. Here we use the ergodicity of the geodesic flow.

Proof. This can be deduced by the work of Basmajian and Šarić [BŠ19, Theorem 3.4 and Proposition 3.7]; if X is not of the first kind, then the convex core is not geodesically complete, and it is possible to attach funnels or half planes to make it geodesically complete. The family of geodesics that escape to infinity through these funnels or half planes gives an open invariant subset of X for the geodesic flow $T^1(X)$. Therefore the geodesic flow is not ergodic, contradicting parabolicity by Theorem 4.3.3. $\square$

The converse is not true in general. An example of a non-parabolic first kind flute surface can be constructed by starting with the interior of a disk and removing countably infinitely many isolated points accumulating onto the whole boundary of the disk. See [GL25] for a detailed construction.

However, there are conditions under which a tight flute surface is of the first kind if and only if it is parabolic.

Theorem 4.3.5 ([PŠ23]). *Let X be a tight flute surface with twist parameters $t_n \in \{0, 1/2\}$ for all n . Then the following are equivalent:*

- X is parabolic;
- X is of the first kind.

Then, by applying Corollary 4.2.4, we obtain a characterization of parabolic tight flute surface with twist parameters equal to 0 or $1/2$.

Theorem 4.3.6. *Let $X = \{\ell_n, t_n\}_{n \in \mathbb{N}^*}$ such that for all n we have $t_n \in \{0, \frac{1}{2}\}$. There exist explicit sequences $(u_n)_{n \in \mathbb{N}^*}$ and $(v_n)_{n \in \mathbb{N}^*}$ such that X is parabolic if and only if*

$$\sum_{n=1}^{\infty} \left(e^{-\frac{\ell_{n+1}}{2}} + e^{-\frac{\ell_n}{2}} \right) \cosh(u_n \ell_n + \dots + u_1 \ell_1) = \infty.$$

Moreover, if ℓ_n is an increasing sequence, then X is parabolic if and only if

$$\sum_{n=1}^{\infty} \left(e^{-\frac{\ell_{n+1}}{2}} + e^{-\frac{\ell_n}{2}} \right) e^{-v_{2n+1}(u_n \ell_n + \dots + u_1 \ell_1)} = \infty.$$

Proof. By Theorem 4.3.5 and Corollary 4.2.4, there exists an explicit sequence $(u_n)_{n \in \mathbb{N}^*}$ such that X is parabolic if and only if

$$\sum_{n=1}^{\infty} \left(e^{-\frac{\ell_{n+1}}{2}} + e^{-\frac{\ell_n}{2}} \right) \cosh(u_n \ell_n + \dots + u_1 \ell_1) = \infty. \quad (4.7)$$

As described in the proof of Theorem 4.2.4, the patchwork associated to $(u_n)_{n \in \mathbb{N}^*}$ is defined so that for all $n > 0$, $v_{2n+2} = v_{2n+1}$ and $v_{2n+1} = v_{2n}$ if $t_n = 0$, $v_{2n+1} = -v_{2n}$ if $t_n = \frac{1}{2}$, and we choose P_2^1 as the starting pentagon (i.e. we set $v_3 = 1$).

Now, let us show that (4.7) is equivalent to

$$\sum_{n=1}^{\infty} \left(e^{-\frac{\ell_{n+1}}{2}} + e^{-\frac{\ell_n}{2}} \right) e^{-v_{2n+1}(u_n \ell_n + \dots + u_1 \ell_1)} = \infty.$$

We have $\frac{e^{|x|}}{2} \leq \cosh(x) \leq e^{|x|}$, thus

$$\frac{1}{2} \sum_{n=1}^{\infty} \ell(\eta_n) e^{|u_n \ell_n + \dots + u_1 \ell_1|} \leq \sum_{n=1}^{\infty} \ell(\eta_n) \cosh(u_n \ell_n + \dots + u_1 \ell_1) \leq \sum_{n=1}^{\infty} \ell(\eta_n) e^{|u_n \ell_n + \dots + u_1 \ell_1|}.$$

Thus

$$\sum_{n=1}^{\infty} \left(e^{-\frac{\ell_{n+1}}{2}} + e^{-\frac{\ell_n}{2}} \right) e^{|u_n \ell_n + \dots + u_1 \ell_1|} = \infty$$

if and only if

$$\sum_{n=1}^{\infty} \left(e^{-\frac{\ell_{n+1}}{2}} + e^{-\frac{\ell_n}{2}} \right) \cosh(u_n \ell_n + \dots + u_1 \ell_1) = \infty.$$

To conclude, we just need to show that we have $|u_n \ell_n + \dots + u_1 \ell_1| = -v_{2n+1}(u_n \ell_n + \dots + u_1 \ell_1)$. To do so we build (by induction) a sequence β_n with a geometric interpretation implying that $\beta_n \geq 0$. Then we show that $\beta_n = -v_{2n+1}(u_n \ell_n + \dots + u_1 \ell_1)$.

Set $\beta_1 = -v_3 u_1 \ell_1$. By definition of u_n , either $t_1 = 0$ and $u_1 = 0$, or $t_1 = \frac{1}{2}$ and $u_1 = -\frac{1}{2}$. Thus $\beta_1 \geq 0$.

Consider the front side of X in Figure 4.22. In the pentagon P_2^1 , we consider a Saccheri quadrilateral with base η_1 , whose one side lies in α_1 and the other side lies in α_2 . Set the length of the side of the Saccheri quadrilateral to be β_1 . Then by induction, we construct a Saccheri quadrilateral in each pentagon $P_{n+1}^{v_{2n+1}}$, with base η_n , whose sides lie in α_n and α_{n+1} such that union of the tops of all Saccheri quadrilaterals forms a continuous path, see Figure 4.22. This path is well defined because $(\ell_n)_{n \in \mathbb{N}^*}$ is an increasing sequence. We define β_n as the length of the side of the Saccheri quadrilateral in $P_{n+1}^{v_{2n+1}}$. Let us compute β_{n+1} in terms of v_n , ℓ_n and u_n . In Figure 4.22, we represent each possible configuration.

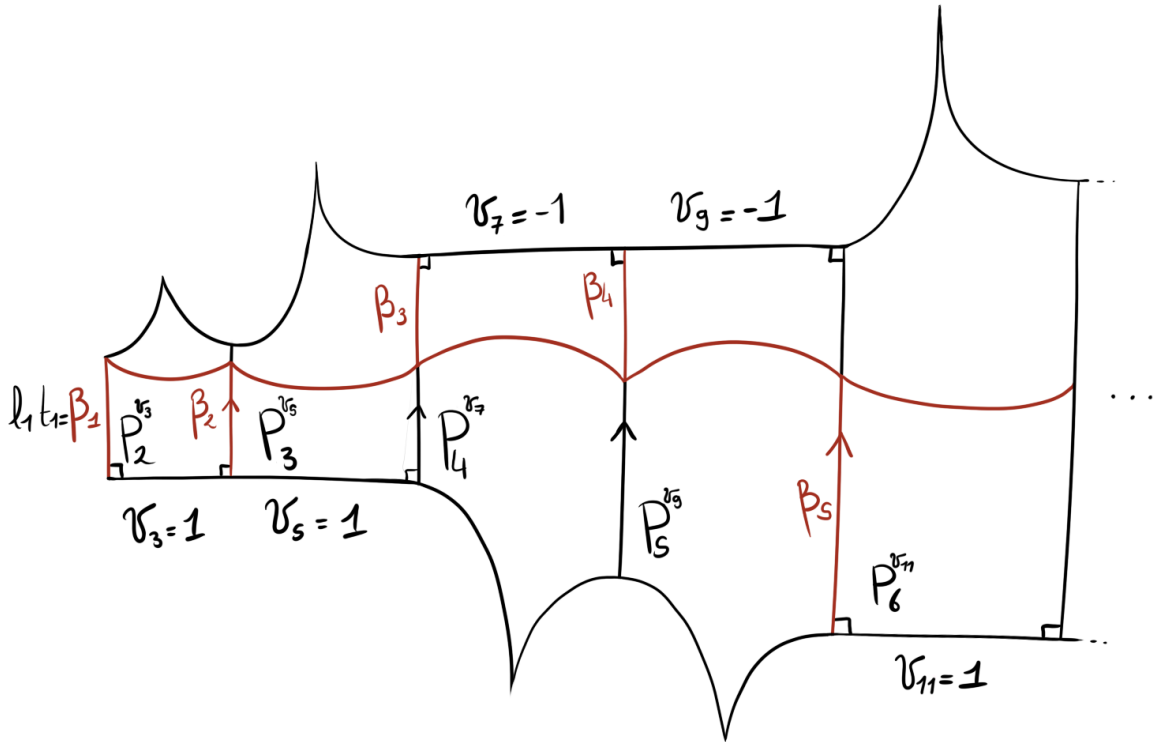


Figure 4.22: Front side of X and construction of β_n .

If $v_{2n+1}v_{2n+3} = 1$, then $t_{n+1} = 0$, thus

$$\beta_{n+1} = \beta_n = v_{2n+1}v_{2n+3}(\beta_n - v_{2n+1}\ell_{n+1}u_{n+1}).$$

If $v_{2n+1}v_{2n+3} = -1$ and $v_{2n+1}t_{n+1} \geq 0$, then $v_{2n+1} = 1$, $v_{2n+3} = -1$ and $t_{n+1} = \frac{1}{2}$. Thus

$$\beta_{n+1} = \frac{\ell_{n+1}}{2} - \beta_n = v_{2n+1}v_{2n+3}(\beta_n - v_{2n+1}\ell_{n+1}u_{n+1}).$$

If $v_{2n+1}v_{2n+3} = -1$ and $v_{2n+1}t_{n+1} < 0$, then $v_{2n+1} = -1$, $v_{2n+3} = 1$ and $t_{n+1} = \frac{1}{2}$. Thus

$$\beta_{n+1} = \frac{\ell_{n+1}}{2} - \beta_n = v_{2n+1}v_{2n+3}(\beta_n - v_{2n+1}\ell_{n+1}u_{n+1}).$$

So in the general case, we have

$$\begin{aligned} \beta_{n+1} &= v_{2n+1}v_{2n+3}(\beta_n - v_{2n+1}\ell_{n+1}u_{n+1}) \\ &= v_{2n+1}v_{2n+3}(-v_{2n+1}\ell_{n+1}u_{n+1} + v_{2n-1}v_{2n+1}(\beta_{n-1} - v_{2n-1}\ell_n u_n)) \\ &= v_{2n+1}v_{2n+3}(-v_{2n+1}\ell_{n+1}u_{n+1} - v_{2n+1}\ell_n u_n + v_{2n+1}v_{2n-3}(\beta_{n-2} - v_{2n-3}\ell_{n-1}u_{n-1})) \\ &= v_{2n+3}(-\ell_{n+1}u_{n+1} - \ell_n u_n - \ell_{n-1}u_{n-1} + v_{2n-3}\beta_{n-2}) \\ &\dots \\ &= v_{2n+3}(-\ell_{n+1}u_{n+1} - \ell_n u_n - \dots + v_3\beta_1) \\ &= v_{2n+3}(-\ell_{n+1}u_{n+1} - \ell_n u_n - \dots - \ell_1 u_1) \\ &= -v_{2n+3}(\ell_{n+1}u_{n+1} + \ell_n u_n + \dots + \ell_1 u_1). \end{aligned}$$

By construction, for all $n > 0$ we have $\beta_n \geq 0$, thus $-v_{2n+1}(\ell_n u_n + \dots + \ell_1 u_1) = |\ell_n u_n + \dots + \ell_1 u_1|$. $\square$

This extends a result of Basmajian, Hakobyan and Šarić [BHŠ22] in the case where all the twists are 0 and a result of Pandazis and Šarić [PŠ23] in the case where all the twists are $\frac{1}{2}$. In particular, with the second sum of the theorem, we can easily retrieve the other two results.

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